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On Semi Weak Factorization Structures

A. Ilaghi-Hosseini, S.Sh. Mousavi, and S.N. Hosseini

Abstract. In this article the notions of semi weak orthogonality and semi weak factorization structure in a category $\mathcal{X}$ are introduced. Then the relationship between semi weak factorization structures and quasi right (left) and weak factorization structures is given. The main result is a characterization of semi weak orthogonality, factorization of morphisms, and semi weak factorization structures by natural isomorphisms.

1 Introduction

The notions of (right, left) factorization structure appeared in [2], while weak factorization structures introduced in [1]. In [9] and [7] the notions of quasi right, respectively, quasi left, factorization structure and some related results has been given. Since in various categories, there are important classes of morphisms that are not factorization structures nor even weak factorization structures, a weaker notion of factorization structure is deemed necessary; so the notion of semi weak factorization structure is introduced. The other main result is to look at semi weak factorization structures as certain isomorphisms in a particular quasicategory.

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In the present article we first give the preliminaries in the current section, as well as a characterization of weak factorization structures in Proposition 1.3. Then in Section 2, we give the notions of semi weak orthogonality and semi weak factorization structure and its relation with quasi right, quasi left, and weak factorization structures. A characterization of semi weak factorization structures is given in Proposition 2.9. We also prove when for a given quasi right structure $\mathcal{M}$, there is an $\mathcal{E}$, with $(\mathcal{E}, \mathcal{M})$ a semi weak factorization structure. In Section 3, we present a characterization of semi weak orthogonality, factorization of morphisms, and semi weak factorization structures in terms of certain natural isomorphisms. Finally in the last section, that is, Section 4, we present several examples of semi weak factorization structures that are not weak factorization structures.

Definition 1.1. See [1] and [4]. Let $\mathcal{E}$ and $\mathcal{M}$ be classes of morphisms in $\mathcal{X}$. We say that $\mathcal{E}$ is (*weakly orthogonal*) *orthogonal* to $\mathcal{M}$, denoted by $(\mathcal{E} \perp^w \mathcal{M})$ $\mathcal{E} \perp \mathcal{M}$, whenever for every commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{u} & M \\ {\scriptstyle e}\downarrow & /// & \downarrow{\scriptstyle m} \\ Y & \xrightarrow[v]{} & Z \end{array}$$

with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, there is a (morphism) uniquely determined morphism $w : Y \longrightarrow M$ with $we = u$ and $mw = v$.

Definition 1.2. [1] A *weak factorization system* in $\mathcal{X}$ is a pair $(\mathcal{E}, \mathcal{M})$ of classes of morphisms such that

(i) $\mathcal{M} = \mathcal{E}^{\square}$ and $\mathcal{E} = {}^{\square}\mathcal{M}$;

(ii) every morphism $f \in \mathcal{X}$ has a factorization $f = me$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$.

Let $\mathcal{E}$ and $\mathcal{M}$ be two classes of morphisms in the category $\mathcal{X}$. Let $e, e' \in \mathcal{E}$ and $m, m' \in \mathcal{M}$ be given. We denote $e \overset{e'}{\square} m$ and $e \underset{m'}{\square} m$, whenever for any f and g making the squares

$$\begin{array}{ccc} \cdot & \xrightarrow{e'} & \cdot \\ {\scriptstyle e}\downarrow & \overset{d'}{\nearrow} & \downarrow{\scriptstyle m} \\ \cdot & \xrightarrow[g]{} & \cdot \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \cdot & \xrightarrow{f} & \cdot \\ {\scriptstyle e}\downarrow & \overset{d}{\nearrow} & \downarrow{\scriptstyle m} \\ \cdot & \xrightarrow[m']{} & \cdot \end{array}$$

commute, there exist diagonals rendering both triangles commutative.

Now we can define the classes $\mathcal{E}^{\square}_{\mathcal{M}}$ and ${}^{\mathcal{E}}_{\square}\mathcal{M}$ as

$\mathcal{E}^{\square}_{\mathcal{M}} = \{m \in \mathcal{M} \mid e \underset{m'}{\square} m, \ \forall e \in \mathcal{E} \text{ and } \forall m' \in \mathcal{M}\}$;

${}^{\mathcal{E}}_{\square}\mathcal{M} = \{e \in \mathcal{E} \mid e \overset{e'}{\square} m, \ \forall e' \in \mathcal{E} \text{ and } \forall m \in \mathcal{M}\}$.

Proposition 1.3. *Let $\mathcal{E}$ and $\mathcal{M}$ be classes of morphisms in $\mathcal{X}$ which are closed under composition. $(\mathcal{E}, \mathcal{M})$ is a weak factorization system if and only if*

(i) $\mathcal{M} = \mathcal{E}^{\square}_{\mathcal{M}}$ *and* $\mathcal{E} = {}^{\mathcal{E}}_{\square}\mathcal{M}$,

(ii) *every morphism $f \in \mathcal{X}$ has a factorization $f = me$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$.*

Proof. This follows directly from the definition. □

With $\langle g \rangle_{\mathcal{E}} = \{ge | ge$ is defined and $e \in \mathcal{E}\}$ (for $\mathcal{E}$ the class of all morphisms, $\langle g \rangle_{\mathcal{E}}$ is just a principal sieve, see [7]), we have

Definition 1.4. See [9]. Suppose that $\mathcal{M}$ is a class of morphisms in $\mathcal{X}$. We say that $\mathcal{X}$ has *quasi right $\mathcal{M}$- factorizations* or $\mathcal{M}$ is a *quasi right factorization structure* in $\mathcal{X}$, whenever for all morphisms $Y \xrightarrow{f} X$ in $\mathcal{X}$, there exists $M \xrightarrow{m_f} X \ \in \mathcal{M}/X$ such that

(a) $\langle f \rangle \subseteq \langle m_f \rangle$;

(b) if $\langle f \rangle \subseteq \langle m \rangle$, with $m \in \mathcal{M}/X$, then $\langle m_f \rangle \subseteq \langle m \rangle$.

m_f is called a quasi right part of f.

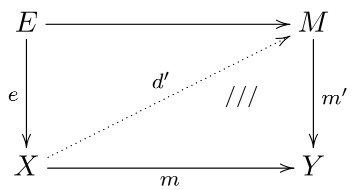
The notion of a cosieve is dual to that of a sieve. A principal cosieve generated by f is denoted by $\rangle f \langle$. Also the notion of a quasi left $\mathcal{E}$-factorization is dual of quasi right factorization, see [7].

2 Semi weak factorization structure

In this section, the notion of semi weak factorization structure, based on semi weak orthogonality, is introduced and its relation with quasi right, quasi left, and weak factorization structures is given. A characterization of semi weak factorization structures is given in Proposition 2.9, which is the counterpart of Proposition 1.3. We also prove when for a given quasi right structure $\mathcal{M}$, there is an $\mathcal{E}$, with $(\mathcal{E}, \mathcal{M})$ a semi weak factorization structure. Some other properties are investigated.

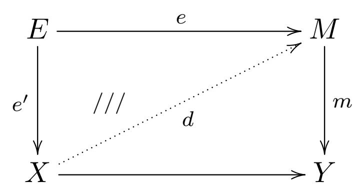
Definition 2.1. Suppose that $\mathcal{X}$ is a category and $\mathcal{E}$ and $\mathcal{M}$ are classes of morphisms in $\mathcal{X}$. We say that $\mathcal{E}$ is *semi weak orthogonal* to $\mathcal{M}$, written $\mathcal{E} \perp^{sw} \mathcal{M}$, whenever

(SW1) for any commutative diagram



where $m, m' \in \mathcal{M}$ and $e \in \mathcal{E}$ there exists a morphism $X \xrightarrow{d'} M$ making the lower triangle commute;

(SW2) for any commutative diagram



where $m \in \mathcal{M}$ and $e, e' \in \mathcal{E}$ there exists a morphism $X \xrightarrow{d} M$ making the upper triangle commute.

Proposition 2.2. *Suppose that $\mathcal{E}$ and $\mathcal{M}$ are classes of morphisms in $\mathcal{X}$. If $\mathcal{E} \perp^{w} \mathcal{M}$, then $\mathcal{E} \perp^{sw} \mathcal{M}$.*

Proof. The proof is straightforward. □

Definition 2.3. Suppose that $\mathcal{X}$ is a category and $\mathcal{E}$ and $\mathcal{M}$ are classes of morphisms in $\mathcal{X}$. We say that $\mathcal{X}$ *has semi weak $(\mathcal{E}, \mathcal{M})$-factorizations* or $(\mathcal{E}, \mathcal{M})$ is a *semi weak factorization structure* in $\mathcal{X}$, whenever

(SWF1) for all $f : Y \longrightarrow X$ there exists $m \in \mathcal{M}/X$ and $e \in Y/\mathcal{E}$ such that $f = me$; and

(SWF2) $\mathcal{E} \perp^{sw} \mathcal{M}$.

Remark 2.4. Weak $(\mathcal{E}, \mathcal{M})$- factorizations are semi weak $(\mathcal{E}, \mathcal{M})$-factorization structures.

Theorem 2.5. *If $\mathcal{X}$ has semi weak $(\mathcal{E}, \mathcal{M})$-factorizations, then $\mathcal{X}$ has quasi right $\mathcal{M}$-factorizations and quasi left $\mathcal{E}$-factorizations.*

Proof. To show that $\mathcal{X}$ has quasi right $\mathcal{M}$-factorizations, let the morphism f in $\mathcal{X}$ be given. By (SWF1), there exist $m_f \in \mathcal{M}$ and $e \in \mathcal{E}$ such that $f = m_f e$. So f factors through m_f. Now suppose that there exist $m \in \mathcal{M}$ such that $\langle f \rangle \subseteq \langle m \rangle$. Thus $\langle m_f e \rangle \subseteq \langle m \rangle$ and so by (SWF2), we have $\langle m_f \rangle \subseteq \langle m \rangle$. Therefore $\mathcal{X}$ has quasi right $\mathcal{M}$-factorizations. Similarly $\mathcal{X}$ has quasi left $\mathcal{E}$-factorizations. $\square$

Corollary 2.6. *If $\mathcal{X}$ has semi weak $(\mathcal{E}, \mathcal{M})$-factorizations and $f = me$, then m is a quasi right part and e is a quasi left part of f.*

Proof. By the fact that $\mathcal{E} \perp^{sw} \mathcal{M}$, the proof is obvious. $\square$

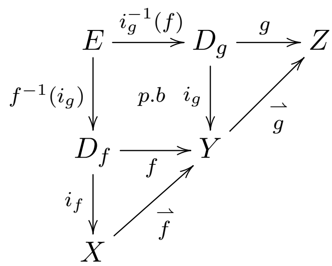
Let $\mathcal{X}$ have pullbacks. The partial morphism category $\vec{\mathcal{X}}$ has the same objects as $\mathcal{X}$, with morphisms $\vec{f} = [(i_f, f)] : X \longrightarrow Y$ equivalence classes of pairs ($i_f : D_f \longrightarrow X$, $f : D_f \longrightarrow Y$) as shown in the following diagram

$$\begin{array}{ccc} D_f & \xrightarrow{f} & Y \\ {\scriptstyle i_f}\downarrow & \nearrow_{\vec{f}} & \\ X & & \end{array}$$

where i_f is a universal mono, that is, its pullback along every morphism exists. Equivalence of (i_f, f) and (i_g, g) means that there is an isomorphism k for which $i_f = i_g k$ and $f = gk$. The composition of morphisms $X \xrightarrow{\vec{f}} Y \xrightarrow{\vec{g}} Z$ is defined by

$$\vec{g}\,\vec{f} = [(i_g, g)][(i_f, f)] = [(i_f \circ f^{-1}(i_g), g \circ i_g^{-1}(f))],$$

as shown in the following diagram



where the commutative square is a pullback square.

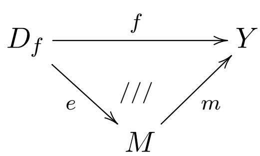
Now let $\mathcal{E}$ and $\mathcal{M}$ be classes of morphisms in $\mathcal{X}$ and $\mathcal{E}'$ and $\mathcal{M}'$ be the classes:

$$\mathcal{E}' = \{[(i_e, e)] \mid \mathrm{dom}(i_e) = \mathrm{dom}\ e,\ e \in \mathcal{E} \quad \text{and} \quad i_e \text{ is a universal mono } \}$$
$$\mathcal{M}' = \{[(1, m)] \mid \mathrm{dom}\ 1 = \mathrm{dom}\ m \text{ and } m \in \mathcal{M}\}$$

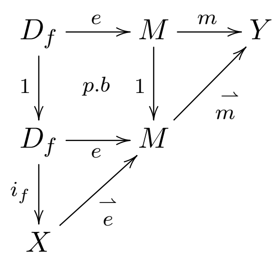
We have the following proposition.

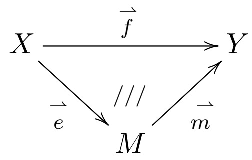
Proposition 2.7. *Let* $(\mathcal{E}, \mathcal{M})$ *be a semi weak factorization structure for* $\mathcal{X}$ *and* $\mathcal{E}$ *be stable under pullbacks, see* [2, Definition 28.13]. *Then* $(\mathcal{E}', \mathcal{M}')$ *is a semi weak factorization structure for* $\vec{\mathcal{X}}$ *.*

Proof. For an arbitrary morphism $\vec{f} = [(i_f, f)] : X \longrightarrow Y$ in $\vec{X}$, since $f \in \mathcal{X}$, we have the commutative diagram



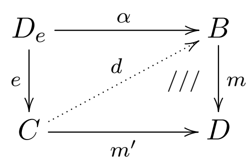
where $e \in \mathcal{E}$ and $m \in \mathcal{M}$. So we have



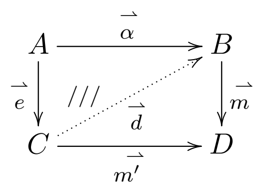


where $\vec{m} = [(1, m)] \in \mathcal{M}'$ and $\vec{e} = [(i_f, e)] \in \mathcal{E}'$.

Suppose that $\vec{e} = [(i_e, e)] \in \mathcal{E}'$ and $\vec{m}, \vec{m'} \in \mathcal{M}'$ are given such that $\langle \vec{m'}\vec{e} \rangle \subseteq \langle \vec{m} \rangle$. Thus $\vec{m'}\vec{e} = \vec{m}\vec{\alpha}$, hence $[(i_\alpha, m\alpha)] = [(i_e, m'e)]$. Since $(\mathcal{E}, \mathcal{M})$ is a semi weak factorization structure for $\mathcal{X}$, we have the diagram

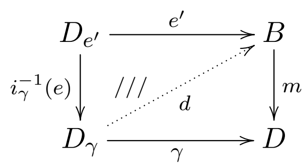


with a diagonal d. Now $\vec{d} = [(1, d)]$ gives the diagonal for the diagram

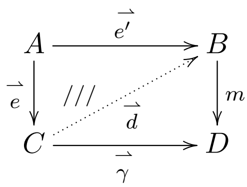


So the condition (SW1) holds.

Now suppose that $\vec{e}, \vec{e'} \in \mathcal{E}'$ and $\vec{m} \in \mathcal{M}'$ are given such that $\rangle \vec{m}\vec{e'} \langle \subseteq \rangle \vec{e} \langle$. Thus $\vec{m}\vec{e'} = \vec{\gamma}\vec{e}$. So, $[(i_e \circ e^{-1}(i_\gamma), \gamma \circ i_\gamma^{-1}(e))] = [(i_{e'}, m \circ e')]$. Since $\mathcal{E}$ is stable under pullbacks, $i_\gamma^{-1}(e) \in \mathcal{E}$. So we have the commutative diagram

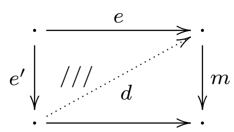


in X, with the diagonal d. Setting $\vec{d} = [(i_\gamma, d)]$, we have the diagram

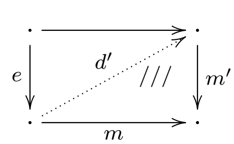


So the condition (SW2) holds. Therefore, $(\mathcal{E}', \mathcal{M}')$ is a semi weak factorization structure for $\vec{\mathcal{X}}$. □

Let $\mathcal{E}$ and $\mathcal{M}$ be two classes of morphisms in the category $\mathcal{X}$, with $e, e' \in \mathcal{E}$ and $m, m' \in \mathcal{M}$. We write $e' \overset{e}{\triangledown} m$ and $e \underset{m}{\triangle} m'$, respectively, whenever in the unbroken commutative diagrams



and



there exist morphisms d and d' such that $e = de'$ and $m = m'd'$.

Remark 2.8. (i) $e' \overset{e}{\triangledown} m$ if and only if, whenever $\rangle me \langle \subseteq \rangle e' \langle$, then $\rangle e \langle \subseteq \rangle e' \langle$;
(ii) $e \underset{m}{\triangle} m'$ if and only if, whenever $\langle me \rangle \subseteq \langle m' \rangle$, then $\langle m \rangle \subseteq \langle m' \rangle$.

Now we can define the classes $\overset{\mathcal{E}}{\triangledown}\mathcal{M}$ and $\mathcal{E}^{\underset{\mathcal{M}}{\triangle}}$ as follows:

$\overset{\mathcal{E}}{\triangledown}\mathcal{M} = \{e' \in \mathcal{E} \mid e' \overset{e}{\triangledown} m, \ \forall e \in \mathcal{E} \text{ and } \forall m \in \mathcal{M}\}$;

$\mathcal{E}^{\underset{\mathcal{M}}{\triangle}} = \{m' \in \mathcal{M} \mid e \underset{m}{\triangle} m', \ \forall e \in \mathcal{E} \text{ and } \forall m \in \mathcal{M}\}$.

Proposition 2.9. *Suppose that $\mathcal{E}$ and $\mathcal{M}$ are classes of morphisms in $\mathcal{X}$. $(\mathcal{E}, \mathcal{M})$ is a semi weak factorization structure in $\mathcal{X}$ if and only if*

(i) $\mathcal{E} = \overset{\mathcal{E}}{\triangledown}\mathcal{M}$ *and* $\mathcal{M} = \mathcal{E}^{\underset{\mathcal{M}}{\triangle}}$;

(ii) *every morphism $f \in \mathcal{X}$ has a factorization $f = me$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$.*

Proof. This follows directly from the definition. □

Lemma 2.10. *Let $(\mathcal{E}, \mathcal{M})$ be a semi weak factorization structure for the category $\mathcal{X}$ then:*

(1) *For any section s, $sf \in \mathcal{E}$ implies that, $f \in \mathcal{E}$.*

(2) *For any retraction r, $fr \in \mathcal{M}$ implies that $f \in \mathcal{M}$.*

Proof. (1) Suppose that $s : C \longrightarrow K$ is a section and $sf \in \mathcal{E}$. By Proposition 2.9 it is enough to show that $f \in \overset{\mathcal{E}}{\triangledown}\mathcal{M}$. Consider the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ {\scriptstyle f}\downarrow & /// & \downarrow{\scriptstyle m} \\ C & \xrightarrow[h]{} & D \end{array}$$

where $m \in \mathcal{M}$ and $e \in \mathcal{E}$. Since s is a section, there exists a morphism $r : K \longrightarrow C$ such that $rs = 1$ and since $sf \in \mathcal{E}$, the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ {\scriptstyle sf}\downarrow & /// \quad \overset{d}{\nearrow} & \downarrow{\scriptstyle m} \\ K & \xrightarrow[hr]{} & D \end{array}$$

has a diagonal d. Hence, ds is the diagonal of the first diagram making the upper triangle commute. Therefore, $f \in \overset{\mathcal{E}}{\triangledown}\mathcal{M}$.

(2) The proof is similar to (1). □

Theorem 2.11. *Let $\mathcal{M}$ be a class of morphisms in $\mathcal{X}$ which is closed under composition and is a quasi right factorization structure. If the class $\mathcal{E} = \{e | \exists f \ni : f = me$ with m a right part of $f\}$ is closed under composition and for all m_1, m_2 in $\mathcal{M}$ and e_1, e_2 in $\mathcal{E}$, the equality $m_1e_1 = m_2e_2$ implies that $\langle m_1 \rangle = \langle m_2 \rangle$ and $\rangle e_1 \langle = \rangle e_2 \langle$, then $(\mathcal{E}, \mathcal{M})$ is a semi weak factorization structure for $\mathcal{X}$.*

Proof. Since $\mathcal{X}$ has quasi right $\mathcal{M}$-factorizations, for all morphisms f in $\mathcal{X}$ there exists $m_f \in \mathcal{M}$ such that $f = m_f e_1$. By definition of $\mathcal{E}$ we have $e_1 \in \mathcal{E}$. To show (SW1), suppose that m and m' in $\mathcal{M}$ and e in $\mathcal{E}$ are given such that $\langle me \rangle \subseteq \langle m' \rangle$. Thus $me = m'g$, and so $me = m'm_g e_2$. Since $\mathcal{M}$ is closed under composition and e and e_2 are in $\mathcal{E}$, $\langle m \rangle = \langle m' m_g \rangle$. Therefore $\langle m \rangle \subseteq \langle m' \rangle$. The Condition (SW2) is proved similarly. Hence $\mathcal{E} \perp^{sw} \mathcal{M}$ and so $\mathcal{X}$ has semi weak $(\mathcal{E}, \mathcal{M})$-factorizations. $\square$

It is known (see [4]) that, for a right factorization $\mathcal{M}$, which is closed under composition, there is an $\mathcal{E}$ such that $(\mathcal{E}, \mathcal{M})$ is a factorization structure. However, for a quasi right factorization $\mathcal{M}$, which is closed under composition, in general, there is no $\mathcal{E}$ such that $(\mathcal{E}, \mathcal{M})$ is a semi weak factorization structure.

Example 2.12. Let $\mathcal{X}$ be the category consisting of the following objects and morphisms only:

$$\mathcal{X}: \; A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{\quad h \quad} C$$

with $hf = hg$. The class $\mathcal{M} = \{1_A, 1_B, 1_C, h\}$, is closed under composition and $\mathrm{Iso}(\mathcal{X}) \subseteq \mathcal{M}$. It is easy to see that $\mathcal{X}$ has quasi right $\mathcal{M}$-factorizations. We claim that there is not a class $\mathcal{E}$ of morphisms in $\mathcal{X}$ such that $\mathcal{X}$ has semi weak $(\mathcal{E}, \mathcal{M})$-factorizations. Otherwise, if $\mathcal{E}$ is such a class, since f can be factored as $f = 1_B f$, with $1_B \in \mathcal{M}$, f must be in $\mathcal{E}$. Similarly, $g \in \mathcal{E}$. On the other hand, $\rangle hf \langle = \rangle hg \langle \subseteq \rangle g \langle$ and $h \in \mathcal{M}$. Since $\mathcal{E} \perp^{sw} \mathcal{M}$, then $\rangle f \langle \subseteq \rangle g \langle$, which is a contradiction.

3 A characterization of semi weak factorization structure

In [9] a one to one correspondence between certain classes of quasi right factorization structures and 2-reflective subobjects of a predefined object in the category of laxed preordered valued presheaves, $\boldsymbol{Lax}(PrOrd^{\boldsymbol{\mathcal{X}}^{op}})$, has been studied. In [7] it has been shown that quasi left factorization structures correspond to subobjects of predefined objects in $\boldsymbol{Lax}(PrOrd^{\boldsymbol{\mathcal{X}}^{op}})$. It has further been shown that this correspondence is one to one when quasi left factorization structures are restricted to the so called QLF-codomains. In this section we give a characterization of semi weak orthogonality, factorization of morphisms, and semi weak factorization structure in a category $\mathcal{X}$ in terms of certain natural isomorphisms.

Denote by $\boldsymbol{PrOrd}$ the category of preordered sets and order preserving functions. Throughout this section suppose that $\mathcal{X}$ has pullbacks and $\mathbf{P}^s : \mathcal{X}^{op} \longrightarrow \boldsymbol{PrOrd}$ the functor defined by

$$\begin{array}{ccc} X & \longmapsto & \mathbf{P}^s(X) = (\{\langle g\rangle | g \in \mathcal{X}/X\}, \subseteq) \\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle \mathbf{P}^s(f)} \\ Y & \longmapsto & \mathbf{P}^s(Y) = (\{\langle h\rangle | h \in \mathcal{X}/Y\}, \subseteq) \end{array}$$

where $\mathbf{P}^s(f)(\langle h\rangle) = \langle h^*_f\rangle$ and h^*_f is a pullback of h along f, and $\mathbf{P}^c : \mathcal{X}^{op} \longrightarrow \boldsymbol{PrOrd}$ the functor defined by

$$\begin{array}{ccc} X & \longmapsto & \mathbf{P}^c(X) = (\{\rangle g\langle | g \in X/\mathcal{X}\}, \subseteq) \\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle \mathbf{P}^c(f)} \\ Y & \longmapsto & \mathbf{P}^c(Y) = (\{\rangle h\langle | h \in Y/\mathcal{X}\}, \subseteq) \end{array}$$

where $\mathbf{P}^c(f)(\rangle h\langle) = (\rangle hf\langle)$.

In what follows $\mathcal{X}_2$ is the class $\{(f,g)|\text{codm}(f) = \text{dom}(g)\}$ of all composable morphisms in $\mathcal{X}$.

Definition 3.1. For a class $\mathcal{A}$ of morphisms in $\mathcal{X}$ we say $\mathcal{A}$ satisfies *the pullback condition*, if for every cospan f and g, with g in $\mathcal{A}$, there exists a pullback of g along f belonging to $\mathcal{A}$.

The following example shows the existence of a class of morphisms that satisfies the pullback condition, which is not stable under pullbacks.

Example 3.2. (i) Let $\mathcal{C}$ be a category with finite products and $\mathcal{M}$ be the class of all pr_2, the second factor projection. Then $\mathcal{M}$ satisfies the pullback condition. To prove this let $f : A \longrightarrow B$ in $\mathcal{C}$ and $pr_2 : C \times B \longrightarrow B$ in $\mathcal{M}$ be given. It is easy to see that the diagram

$$\begin{array}{ccc} C \times A & \xrightarrow{1_C \times f} & C \times B \\ {\scriptstyle pr_2}\downarrow & & \downarrow{\scriptstyle pr_2} \\ A & \xrightarrow{\quad f \quad} & B \end{array}$$

is a pullback.

(ii) Let $\mathfrak{K}$ be a field, R be a finite dimensional $\mathfrak{K}$-algebra and $\mathcal{C}$ be a full subcategory of **RMod**, of left R-modules, whose objects are finitely generated R-modules. Let $\mathcal{M}$ be the class of all morphisms in $\mathcal{C}$ which factors through a non isomorphism pr_2. Then $\mathcal{M}$ satisfies the pullback condition. To prove this let $f : M \longrightarrow K$ in $\mathcal{C}$ and $m : N \times L \longrightarrow K$ in $\mathcal{M}$ be given. Thus $m = gpr_2$ such that $pr_2 : N \times L \longrightarrow L$ is a non isomorphism. By (i) we have the pullback diagram

$$\begin{array}{ccc} N \times P & \xrightarrow{1_N \times g^{-1}(f)} & N \times L \\ {\scriptstyle pr'_2}\downarrow & & \downarrow{\scriptstyle pr_2} \\ P & \xrightarrow{g^{-1}(f)} & L \\ {\scriptstyle f^{-1}(g)}\downarrow & & \downarrow{\scriptstyle g} \\ M & \xrightarrow{f} & K \end{array}$$

The morphism pr'_2 is not an isomorphism, because otherwise, since every module is finite dimensional as a $\mathfrak{K}$-vector spaces, $\dim_{\mathfrak{K}} N = 0$ and so $N = 0$. Therefore pr_2 is an isomorphism, and this is a contradiction.

First, we characterize semi weak orthogonality in two steps.

Step 1: Suppose $\mathcal{E}$ and $\mathcal{M}$ are classes of morphisms in $\mathcal{X}$ which satisfy the pullback condition. Define the equivalence relation $\sim_s$ on $\mathcal{X}_2$ as

$$(f, g) \sim_s (f', g') \text{ if and only if } \langle gf \rangle = \langle g'f' \rangle \text{ and } \langle g \rangle = \langle g' \rangle.$$

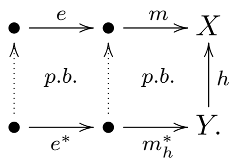
Define the functor $\mathbf{R}_s : \mathcal{X}^{op} \longrightarrow \boldsymbol{PrOrd}$ by

$$\begin{array}{cccc} Y & \longmapsto & \mathbf{R}_s(Y) = (\{[(e', m')]_s | e' \in \mathcal{E}, m' \in \mathcal{M}/Y, (e', m') \in \mathcal{X}_2\}, \leqslant_s) & \\ {\scriptstyle h}\downarrow & & \uparrow{\scriptstyle \mathbf{R}_s(h)} & \\ X & \longmapsto & \mathbf{R}_s(X) = (\{[(e, m)]_s | e \in \mathcal{E}, m \in \mathcal{M}/X, (e, m) \in \mathcal{X}_2\}, \leqslant_s) & \end{array}$$

where the relation $\leqslant_s$ is defined as

$$[(e, m)]_s \leqslant_s [(e', m')]_s \text{ if and only if } \langle me \rangle \subseteq \langle m'e' \rangle \text{ and } \langle m \rangle \subseteq \langle m' \rangle;$$

and $\mathbf{R}_s(h)([(e,m)]_s) = [(e^*, m_h^*)]_s$, where m_h^* and e^* are obtained by the following pullback diagrams



Also the family $\boldsymbol{r}_s : \mathbf{R}_s \longrightarrow \mathbf{P}^s$ defined, for each X, by

$$(\boldsymbol{r}_s)_X([(e,m)]_s) = \langle me \rangle$$

can be easily verified to be a natural transformation.

Lemma 3.3. *Suppose that $\mathcal{E}$ and $\mathcal{M}$ are classes of morphisms in $\mathcal{X}$ that satisfy the pullback condition, $\mathcal{M}$ is closed under composition and for all morphisms f in $\mathcal{X}$ there exist morphisms $m_f \in \mathcal{M}$ and $e_f \in \mathcal{E}$ such that $f = m_f e_f$. Then for all $m, m' \in \mathcal{M}$ and for all $e \in \mathcal{E}$, $\langle me \rangle \subseteq \langle m' \rangle$ implies $\langle m \rangle \subseteq \langle m' \rangle$ if and only if $\boldsymbol{r}_s$ is a natural isomorphism.*

Proof. To prove the necessity, the family $\boldsymbol{\alpha} : \mathbf{P}^s \longrightarrow \mathbf{R}_s$ defined, for each X, by $\boldsymbol{\alpha}_X(\langle f \rangle) = [(e_f, m_f)]_s$ is a natural transformation. It is easy to see that $\boldsymbol{\alpha}\boldsymbol{r}_s = \mathbf{1}$ and $\boldsymbol{r}_s\boldsymbol{\alpha} = \mathbf{1}$ so $\boldsymbol{r}_s$ is a natural isomorphism. To prove the sufficiency, suppose that $\boldsymbol{\beta} : \mathbf{P}^s \longrightarrow \mathbf{R}_s$ is the inverse of $\boldsymbol{r}_s$ and $m, m' \in \mathcal{M}$ and $e \in \mathcal{E}$ are given such that $\langle me \rangle \subseteq \langle m' \rangle$. Thus there exists a morphism k such that $me = m'k$ and so $\boldsymbol{r}_s([(e,m)]_s) = \boldsymbol{r}_s([(e_k, m'm_k)]_s)$, where $k = m_k e_k$. Therefore $(e,m) \sim_s (e_k, m'm_k)$ and hence $\langle m \rangle = \langle m'm_k \rangle \subseteq \langle m' \rangle$. $\square$

Step 2: let $\mathcal{E}$ and $\mathcal{M}$ be classes of morphisms in $\mathcal{X}$ such that $\mathcal{M}$ is closed under composition, for all morphisms f in $\mathcal{X}$ there exist morphisms $m_f \in \mathcal{M}$ and $e_f \in \mathcal{E}$ such that $f = m_f e_f$ and we have

(a) for all $g \in \mathcal{E}$, $\rangle e_g \langle = \rangle g \langle$;

(b) $\rangle f \langle \subseteq \rangle g \langle$ implies $\rangle e_f \langle \subseteq \rangle e_g \langle$.

By a similar fashion, define the equivalence relation $\sim_c$ on $\mathcal{X}_2$ as

$$(f,g) \sim_c (f',g') \text{ if and only if } \rangle gf \langle = \rangle g'f' \langle \text{and} \rangle f \langle = \rangle f' \langle$$

and define $\mathbf{R}_c : \mathcal{X}^{op} \longrightarrow \boldsymbol{PrOrd}$ by

$$\begin{array}{ccc}
Y & \longmapsto & \mathbf{R}_c(Y) = (\{[(e', m')]_c \mid e' \in Y/\mathcal{E}, m' \in \mathcal{M}, (e', m') \in \mathcal{X}_2\}, \leqslant_c) \\
{\scriptstyle h}\downarrow & & \uparrow{\scriptstyle \mathbf{R}_c(h)} \\
X & \longmapsto & \mathbf{R}_c(X) = (\{[(e, m)]_c \mid e \in X/\mathcal{E}, m \in \mathcal{M}, (e, m) \in \mathcal{X}_2\}, \leqslant_c)
\end{array}$$

where the relation $\leqslant_c$ is defined as

$$[(e,m)]_c \leqslant_c [(e',m')]_c \text{ if and only if } \rangle me\langle \subseteq \rangle m'e'\langle \text{ and } \rangle e\langle \subseteq \rangle e'\langle;$$

and $\mathbf{R}_c(h)([(e,m)]_c) = [(e_{eh}, m m_{eh})]_c$, where $eh = m_{eh} e_{eh}$ such that $m_{eh} \in \mathcal{M}$ and $e_{eh} \in \mathcal{E}$. It is easy to see that $\mathbf{R}_c$ is a lax functor.

Also the family $\boldsymbol{r}_c : \mathbf{R}_c \longrightarrow \mathbf{P}^c$ defined, for each X, by

$$(\boldsymbol{r}_c)_X([(e,m)]_c) = \rangle me\langle$$

can be easily verified to be a natural transformation.

Lemma 3.4. *Suppose that $\mathcal{E}$ and $\mathcal{M}$ are classes of morphisms in $\mathcal{X}$ which are closed under composition. Also suppose that for all morphisms f in $\mathcal{X}$ there exist morphisms $m_f \in \mathcal{M}$ and $e_f \in \mathcal{E}$ such that $f = m_f e_f$ and we have*

(a) for all $g \in \mathcal{E}$, $\rangle e_g\langle = \rangle g\langle$; and

(b) $\rangle f\langle \subseteq \rangle g\langle$ implies $\rangle e_f\langle \subseteq \rangle e_g\langle$.

Then for all $e, e' \in \mathcal{E}$ and for all $m \in \mathcal{M}$, $\rangle me\langle \subseteq \rangle e'\langle$, implies $\rangle e\langle \subseteq \rangle e'\langle$ if and only if $\boldsymbol{r}_c$ is a natural isomorphism.

Proof. The proof is similar to Lemma 3.3. □

Proposition 3.5. *Let $\mathcal{E}$ and $\mathcal{M}$ be classes of morphisms in $\mathcal{X}$ that satisfy the pullback condition are closed under compositions. Also for all morphisms f in $\mathcal{X}$ there exist morphisms $m_f \in \mathcal{M}$ and $e_f \in \mathcal{E}$ such that $f = m_f e_f$ and we have*

(a) for all $g \in \mathcal{E}$, $\rangle e_g\langle = \rangle g\langle$; and

(b) $\rangle f\langle \subseteq \rangle g\langle$ implies $\rangle e_f\langle \subseteq \rangle e_g\langle$.

Then $\mathcal{E} \perp^{sw} \mathcal{M}$ if and only if $\boldsymbol{r}_s$ and $\boldsymbol{r}_c$ are natural isomorphisms.

Proof. It follows from Lemmas 3.3 and 3.4. □

Next we give a characterization of the factorizations of morphisms. To this end, assume $\mathcal{E}$ and $\mathcal{M}$ are classes of morphisms in $\mathcal{X}$ that satisfy the pullback condition, $\mathcal{E}$ is closed under composition, and $\mathrm{Iso}(\mathcal{X}) \subseteq \mathcal{E}$, where

Iso($\mathcal{X}$) is the class of isomorphisms in $\mathcal{X}$. Suppose that $f \in \mathcal{X}/X$ is given. Since $\mathcal{E}$ is closed under composition, if $g \in \langle f \rangle_{\mathcal{E}}$, then $\langle g \rangle_{\mathcal{E}} \subseteq \langle f \rangle_{\mathcal{E}}$. Define $\mathbf{P}^s_{\mathcal{E}} : \mathcal{X}^{op} \longrightarrow \boldsymbol{PrOrd}$ by

$$\begin{array}{ccl} Y & \longmapsto & \mathbf{P}^s_{\mathcal{E}}(Y) = (\{\langle g \rangle_{\mathcal{E}} | g \in \mathcal{X}/Y\}, \subseteq) \\ {\scriptstyle h}\downarrow & & \quad\uparrow {\scriptstyle \mathbf{P}^s_{\mathcal{E}}(h)} \\ X & \longmapsto & \mathbf{P}^s_{\mathcal{E}}(X) = (\{\langle f \rangle_{\mathcal{E}} | f \in \mathcal{X}/X\}, \subseteq) \end{array}$$

where $\mathbf{P}^s_{\mathcal{E}}(h)(\langle f \rangle_{\mathcal{E}}) = \langle h^*_f \rangle_{\mathcal{E}}$.

Now define the equivalence relation $\sim_{\mathcal{E}}$ on $\mathcal{X}_2$ as follows

$$(f_1, g_1) \sim_{\mathcal{E}} (f_2, g_2) \text{ if and only if } \langle g_1 f_1 \rangle_{\mathcal{E}} = \langle g_2 f_2 \rangle_{\mathcal{E}}.$$

We denote the equivalence class of (f, g) by $[(f, g)]_{\mathcal{E}}$.

Define the functor $\mathbf{P_2}_{\mathcal{E}} : \mathcal{X}^{op} \longrightarrow \boldsymbol{PrOrd}$ as

$$\begin{array}{ccl} Y & \longmapsto & \mathbf{P_2}_{\mathcal{E}}(Y) = (\{[(f_2, g_2)]_{\mathcal{E}} | (f_2, g_2) \in \mathcal{X}_2, g_2 \in \mathcal{X}/Y\}, \leqslant) \\ {\scriptstyle h}\downarrow & & \qquad\qquad\uparrow {\scriptstyle \mathbf{P_2}_{\mathcal{E}}(h)} \\ X & \longmapsto & \mathbf{P_2}_{\mathcal{E}}(X) = (\{[(f_1, g_1)]_{\mathcal{E}} | (f_1, g_1) \in \mathcal{X}_2, g_1 \in \mathcal{X}/X\}, \leqslant) \end{array}$$

where the relation $\leqslant$ is defined by

$$[(f_1, g_1)]_{\mathcal{E}} \leqslant [(f_2, g_2)]_{\mathcal{E}} \text{ if and only if } \langle g_1 f_1 \rangle_{\mathcal{E}} \subseteq \langle g_2 f_2 \rangle_{\mathcal{E}}$$

and $\mathbf{P_2}_{\mathcal{E}}(h)([(f, g)]_{\mathcal{E}}) = [(f^*_h, g^*)]_{\mathcal{E}}$, where f^*_h and g^* are obtained by the pullback diagrams

$$\begin{array}{ccccc} \bullet & \xrightarrow{g} & \bullet & \xrightarrow{f} & X \\ \uparrow & p.b. & \uparrow & p.b. & \uparrow {\scriptstyle h} \\ \bullet & \xrightarrow[g^*]{} & \bullet & \xrightarrow[f^*_h]{} & Y. \end{array}$$

Thus the family $*_{\mathcal{E}} : \mathbf{P_2}_{\mathcal{E}} \stackrel{\cdot}{\longrightarrow} \mathbf{P}^s_{\mathcal{E}}$ defined, for each X, by

$$(*_{\mathcal{E}})_X([(f, g)]_{\mathcal{E}}) = \langle gf \rangle_{\mathcal{E}}$$

is a natural transformation.

Since $\mathcal{E}$ and $\mathcal{M}$ satisfy the pullback condition, $\mathbf{R}_{(\mathcal{E},\mathcal{M})} : \mathcal{X}^{op} \longrightarrow \boldsymbol{PrOrd}$ defined by

$$\begin{array}{ccc} Y & \longmapsto & \mathbf{R}_{(\mathcal{E},\mathcal{M})}(Y) = (\{[(e',m')]_{\mathcal{E}} | (e',m') \in \mathcal{X}_2, e' \in \mathcal{E}, m' \in \mathcal{M}/X\}, \leqslant) \\ {\scriptstyle h}\downarrow & & \uparrow {\scriptstyle \mathbf{R}_{(\mathcal{E},\mathcal{M})}(h)} \\ X & \longmapsto & \mathbf{R}_{(\mathcal{E},\mathcal{M})}(X) = (\{[(e,m)]_{\mathcal{E}} | (e,m) \in \mathcal{X}_2, e \in \mathcal{E}, m \in \mathcal{M}/X\}, \leqslant) \end{array}$$

and $\mathbf{R}_{(\mathcal{E},\mathcal{M})}(h)([(e,m)]_{\mathcal{E}}) = [(m_h^*, e^*)]_{\mathcal{E}}$ is a subfunctor of $\mathbf{P}_{\mathbf{2}_{\mathcal{E}}}$. So we have the natural transformation

$$\mathbf{j}_{(\mathcal{E},\mathcal{M})} : \mathbf{R}_{(\mathcal{E},\mathcal{M})} \overset{\cdot}{\hookrightarrow} \mathbf{P}_{\mathbf{2}_{\mathcal{E}}}.$$

Now define

$$*_{(\mathcal{E},\mathcal{M})} = *_{\mathcal{E}} \ \mathbf{j}_{(\mathcal{E},\mathcal{M})} : \mathbf{R}_{(\mathcal{E},\mathcal{M})} \overset{\cdot}{\longrightarrow} \mathbf{P}^s_{\mathcal{E}}$$

so that $(*_{(\mathcal{E},\mathcal{M})})_X([(e,m)]_{\mathcal{E}}) = \langle me \rangle_{\mathcal{E}}$.

Proposition 3.6. *Suppose $\mathcal{E}$ and $\mathcal{M}$ are classes of morphisms in $\mathcal{X}$ which satisfy the pullback condition, $\mathcal{E}$ is closed under composition, and* $\text{Iso}(\mathcal{X}) \subseteq \mathcal{E}$*. Then for all morphisms $f \in \mathcal{X}$ there exist morphisms $m \in \mathcal{M}$ and $e \in \mathcal{E}$ such that $f = me$ if and only if $*_{\mathcal{E},\mathcal{M}}$ is a natural isomorphism.*

Proof. First suppose that $*_{\mathcal{E},\mathcal{M}}$ is a natural isomorphism. Let $f \in \mathcal{X}/X$ be given. Suppose that $(*_{(\mathcal{E},\mathcal{M})})^{-1}_X(\langle f \rangle_{\mathcal{E}}) = [(e_1, m)]_{\mathcal{E}}$ and so $\langle f \rangle_{\mathcal{E}} = \langle me_1 \rangle_{\mathcal{E}}$. Since $f \in \langle f \rangle_{\mathcal{E}}$, there exists $e_2 \in \mathcal{E}$ such that $f = me_1e_2$. Set $e = e_1e_2$. Since $\mathcal{E}$ is closed under composition, $e \in \mathcal{E}$. Thus $f = me$. For the proof of the converse, suppose for all $f \in \mathcal{X}$ there exist morphisms $e_f \in \mathcal{E}$ and $m_f \in \mathcal{M}$ such that $f = m_f e_f$. The mapping $\boldsymbol{r} : \mathbf{P}^s_{\mathcal{E}} \overset{\cdot}{\longrightarrow} \mathbf{R}_{(\mathcal{E},\mathcal{M})}$ defined for all X in $\mathcal{X}$ by $\boldsymbol{r}_X(\langle f \rangle_{\mathcal{E}}) = [(e_f, m_f)]_{\mathcal{E}}$ is a natural transformation and $*_{(\mathcal{E},\mathcal{M})} \ \boldsymbol{r} = 1_{\mathbf{P}_{\mathcal{E}}}$. It easily follows that $\boldsymbol{r} *_{(\mathcal{E},\mathcal{M})} = 1_{\mathbf{R}_{(\mathcal{E},\mathcal{M})}}$. So $\boldsymbol{r}$ is a natural isomorphism. □

Finally, the following theorem gives a characterization of semi weak factorization structures, under certain conditions.

Theorem 3.7. *Suppose $\mathcal{E}$ and $\mathcal{M}$ are classes of morphisms in $\mathcal{X}$ that satisfy the pullback condition, are closed under composition, and* $\text{Iso}(\mathcal{X}) \subseteq \mathcal{E}$*. Then $(\mathcal{E},\mathcal{M})$ is a semi weak factorization structure in $\mathcal{X}$ if and only if $\boldsymbol{r}_s$, $\boldsymbol{r}_c$, and $*_{\mathcal{E},\mathcal{M}}$ are natural isomorphisms.*

Proof. Follows from propositions 3.5 and 3.6. □

4 Examples

In this section we give several examples of semi weak factorization structures which are not weak factorization structures.

Example 4.1. Consider the category **Set** of sets. Let $\mathcal{E}$ be the class of monomorphisms and $\mathcal{M}$ be defined by

(i) each pr_2, the second factor projection, is in $\mathcal{M}$;

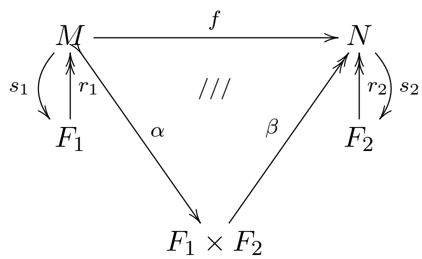
(ii) for each $m_1, m_2 \in \mathcal{M}$, $m_2 m_1 \in \mathcal{M}$, whenever $m_2 m_1$ is defined;

(iii) the class $\mathcal{M}$ is generated by (i) and (ii).

For an arbitrary function $f : X \longrightarrow Y$ we have $f = pr_2\langle 1, f\rangle$. Since $\mathcal{M}$ is in the class of epimorphisms, $\mathcal{E} \perp^w \mathcal{M}$. Since $\operatorname{Iso}\mathcal{X} \nsubseteq \mathcal{M}$, this factorization system is not weak.

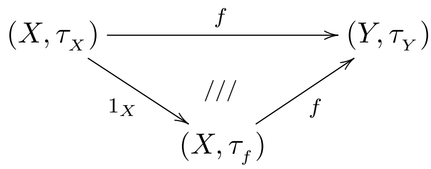
Example 4.2. Consider the category **RMod** such that R is a left semisimple ring. Let $\mathcal{E}$ and $\mathcal{M}$ be as in Example 4.1. For an arbitrary R-module homomorphism $f : X \longrightarrow Y$ we have $f = pr_2\langle 1, f\rangle$. Since every module is injective and projective, $\mathcal{E} \perp^{sw} \mathcal{M}$. Since $\operatorname{Iso}\mathcal{X} \nsubseteq \mathcal{M}$, this factorization system is not weak.

Example 4.3. Let $\mathfrak{K}$ be a field and R be a finite dimensional $\mathfrak{K}$-algebra, left injective R-module, and semihereditary ring, see [12, Definition 39.1]. Let $\mathbb{P}$ be a full subcategory of **RMod** whose objects are finitely generated projective R-modules. Let $\mathcal{E}$ be the class of monomorphisms in $\mathbb{P}$ and $\mathcal{M}$ be the class of those morphisms in $\mathbb{P}$ which factor through a non isomorphism pr_2. For each momorphism $f : M \longrightarrow N$ in $\mathbb{P}$ we have

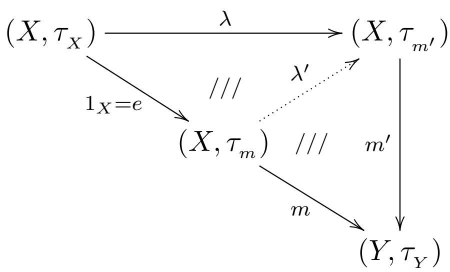


where F_1 and F_2 are finitely generated free R-modules, $r_1 s_1 = 1_M$, $r_2 s_2 = 1_N$, $\alpha = \langle s_1, s_2 f\rangle$, and $\beta = r_2 pr_2$. Thus $f = \beta\alpha$, $\alpha \in \mathcal{E}$ and $\beta \in \mathcal{M}$. Since every object in $\mathbb{P}$ is also injective, $\mathcal{E} \perp^{sw} \mathcal{M}$. Since $\operatorname{Iso}\mathcal{X} \nsubseteq \mathcal{M}$, this factorization system is not weak.

Example 4.4. Consider the category ***Top*** of topological spaces. Let $\mathcal{M}$ be the class of initial maps and $\mathcal{E}$ be the class of all continuous maps with identity as the underlying map. Suppose that $f : X \longrightarrow Y$ in ***Top*** is given. We have



where τ_f is the induced topology by f on X. Now suppose that there is $m, m' \in \mathcal{M}$ and $e \in \mathcal{E}$ such that $\langle me \rangle \subseteq \langle m' \rangle$. So there is a morphism λ such that $me = m'\lambda$. It is easy to see that $\lambda' = \lambda$ in the diagram



makes the triangles commute and is in ***Top***. So $\langle m \rangle \subseteq \langle m' \rangle$. The proof of the second part is similar. Hence $\mathcal{E} \perp^{sw} \mathcal{M}$. Since Iso $\mathcal{X} \nsubseteq \mathcal{E}$, this factorization system is not weak.

Lemma 4.5. *If* $(\mathcal{E}, \mathcal{M})$ *and* $(\mathcal{E}, \mathcal{M}')$ *are weak factorization systems in* $\mathcal{X}$*, then* $\mathcal{M} = \mathcal{M}'$

Proof. Let $m' \in \mathcal{M}'$ be given. Thus, there exist $e \in \mathcal{E}$ and $m \in \mathcal{M}$ such that $m' = me$ and hence there exists a morphism d such that $de = 1$ and $m'd = m$. So d is a retraction and $m'd \in \mathcal{M}$. Therefore [1, Observation 1.3 (2b′)] implies that $m' \in \mathcal{M}$ and hence $\mathcal{M}' \subseteq \mathcal{M}$. Similarly, we have $\mathcal{M} \subseteq \mathcal{M}'$. □

Example 4.6. Let $\mathcal{C}$ be a closed model category whose objects are cofibrant-fibrant. The pair $(\mathcal{E}, \mathcal{M})$ of morphisms in $\mathcal{C}$, where $\mathcal{E}$ is the class of cofibrations and $\mathcal{M}$ is the class of weak equivalences which are retractions form a semi weak factorization structure. Because, by [5, Definition 7.1.3] every morphism f in $\mathcal{C}$ has a factorization $f = pj$, where j is a cofibration and p is a trivial fibration and by [5, Proposition 7.6.11 (2)] p is a retraction. It is easy to see that $\mathcal{M} = \mathcal{E}^{\underset{\mathcal{M}}{\triangle}}$. To prove $\mathcal{E} = {}^{\overset{\mathcal{E}}{\triangledown}}\mathcal{M}$ suppose that in the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ {\scriptstyle i'}\downarrow & /// & \downarrow{\scriptstyle w} \\ C & \xrightarrow[v]{} & D \end{array}$$

$i, i' \in \mathcal{E}$, $w \in \mathcal{M}$, and v is an arbitrary. By [5, Proposition 7.2.6]

$$B \xrightarrow{w} D = B \xrightarrow{j} W \xrightarrow{p'} D$$

where j is a trivial cofibration and p' is a trivial fibration. Let $\mathcal{M}'$ be the class of trivial fibrations. So by [5, Definition 7.1.3], we have $\mathcal{E} \perp^{w} \mathcal{M}'$. Thus, there exists a morphism $C \xrightarrow{d'} W$ such that $p'd' = v$ and $d'i' = ji$. By [5, Proposition 7.6.11 (1)] there exists a morphism $W \xrightarrow{r} B$ such that $rj = 1_B$. Put $d := rd'$, so $di' = i$. Therefore $\mathcal{E} = {}^{\overset{\mathcal{E}}{\triangledown}}\mathcal{M}$ and hence, by Proposition 2.9, we have $(\mathcal{E}, \mathcal{M})$ is a semi weak factorization structure. This factorization system is not weak. Because otherwise, since by [1, Remark 3.6] $(\mathcal{E}, \mathcal{M}')$ is a weak factorization system, Lemma 4.5 implies that $\mathcal{M} = \mathcal{M}'$. However this is not the case, for instance in ***Top*** let $E = \{0\} \times I \bigcup I \times \{0\}$ (with the topology induced by $\mathbb{R}^2$), $B = I \times \{0\}$ and $p : E \longrightarrow B$ the projection on the first factor. Then p is not a fibration, see [10, Exercises 2.2.9]. Therefore $p \notin \mathcal{M}'$ but $p \in \mathcal{M}$.

Example 4.7. Examples 4.1 to 4.3 satisfy the conditions of the Proposition 2.7. Thus in the corresponding partial morphism categories the classes $\mathcal{E}'$ and $\mathcal{M}'$ constitute a semi weak factorization structures which are not weak factorization structures, because otherwise $(\mathcal{E}, \mathcal{M})$ is a weak factorization structure which is a contradiction.

Example 4.8. Let $\mathcal{X}$ be a connected category with coproducts. Then

$$\mathcal{E} = \{\, A \xrightarrow{\nu_1} A \coprod B \;\mid \nu_1 \text{ is a coproduct inclusion to the first factor}\}$$

and $\mathcal{M}$ any collection of retractions constitute a semi weak factorization system. Since $\operatorname{Iso}\mathcal{X} \nsubseteq \mathcal{E}$, this factorization system is not weak.

Example 4.9. Consider the category ***Set*** of sets. Define the classes $\mathcal{E}$ and $\mathcal{M}$ as

$$\mathcal{E} = \{\, X \xrightarrow{\mathfrak{f}} X \times Y \mid \; X \xrightarrow{f} Y \;\text{ is a morphism in } \boldsymbol{Set} \text{ and } \mathfrak{f} = \langle 1, f\rangle\}$$
$$\mathcal{M} = \{\, X \times Y \xrightarrow{\quad pr_2 \quad} Y \mid pr_2 \text{ is the second factor projection}\}.$$

For an arbitrary function $f : X \longrightarrow Y$ we have $f = pr_2\langle 1, f\rangle$. Since $\mathcal{M}$ is in the class of epimorphisms, $\mathcal{E} \perp^w \mathcal{M}$. Since $\operatorname{Iso}\mathcal{X} \nsubseteq \mathcal{M}$, this factorization system is not weak.

Example 4.10. Consider the category ***Ab*** of abelian groups. Define the classes $\mathcal{E}$ and $\mathcal{M}$ as

$$\mathcal{E} = \{\, G \xrightarrow{\mathfrak{e}} G \times G \mid \mathfrak{e}(x) = (x, e), \text{ where } \mathfrak{e} = \langle 1, e\rangle\}$$
$$\mathcal{M} = \{\, G \times G \xrightarrow{f} H \mid f \text{ factors through the operation } \star \text{ of } G\}.$$

For an arbitrary morphism $f : G \longrightarrow H$, we have $f = (f\star)\mathfrak{e}$. Suppose that $f, g \in \mathcal{M}$ and $\mathfrak{e} \in \mathcal{E}$ are given such that $\langle f\mathfrak{e}\rangle \subseteq \langle g\rangle$, thus $f\mathfrak{e} = gh$. So we have

$$\begin{array}{ccc} G & \xrightarrow{h} & K \times K \\ {\scriptstyle \mathfrak{e}}\downarrow & \overset{d}{\nearrow} /// & \downarrow{\scriptstyle g} \\ G \times G & \xrightarrow[f]{} & H \end{array}$$

where $d = h\star$ and $\star$ is the operation of G.

Now suppose that $f \in \mathcal{M}$ and $\mathfrak{e} \in \mathcal{E}$ are given such that $\rangle f\mathfrak{e}\langle \subseteq \rangle\mathfrak{e}\langle$. Thus $f\mathfrak{e} = k\mathfrak{e}$. So we have

$$\begin{array}{ccc} G & \xrightarrow{\mathfrak{e}} & G \times G \\ {\scriptstyle \mathfrak{e}}\downarrow & /// \underset{d}{\nearrow} & \downarrow{\scriptstyle f} \\ G \times G & \xrightarrow[k]{} & H \end{array}$$

where $d = 1_{G\times G}$. Therefore, $(\mathcal{E}, \mathcal{M})$ constitutes a semi weak factorization structure for ***Ab***. Since $\operatorname{Iso}\mathcal{X} \nsubseteq \mathcal{M}$, this factorization system is not weak.

Example 4.11. Let $\mathcal{X}$ be a pointed category, see [2]. Fix a non terminal object $B \in \mathcal{X}$. Define the classes $\mathcal{E}$ and $\mathcal{M}$ as

$$\mathcal{E} = \{\, X \xrightarrow{\mathfrak{f}} Y \times Y \mid X \xrightarrow{f} Y \text{ is a morphism in } \mathcal{X} \text{ and } \mathfrak{f} = \langle f, f \rangle \}$$

$$\mathcal{M} = \{\, X \times A \xrightarrow{pr_1} X \mid X \neq B\} \bigcup \{\, B \times C \xrightarrow{pr_1} B \mid C \text{ is not terminal object}\}$$

where pr_1 is the first factor projection. Every morphism $f : X \longrightarrow Y$ in $\mathcal{X}$ can be factored as $f = pr_1\mathfrak{f}$. To show $(\mathcal{E}, \mathcal{M})$ constitutes a semi weak factorization structure for $\mathcal{X}$, suppose $pr_1, pr'_1 \in \mathcal{M}$ and $\mathfrak{f} \in \mathcal{E}$ are given such that $\langle pr_1\mathfrak{f} \rangle \subseteq \langle pr'_1 \rangle$. Thus $pr_1\mathfrak{f} = pr'_1 u$. So we have

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \times Z \\ {\scriptstyle \mathfrak{f}}\downarrow & \overset{d}{\nearrow} \; /// & \downarrow{\scriptstyle pr'_1} \\ Y \times Y & \xrightarrow[pr_1]{} & Y \end{array}$$

where d is the unique morphism making the following diagram

$$\begin{array}{ccccc} Y & \xleftarrow{pr'_1} & Y \times Z & \xrightarrow{pr'_2} & Z \\ & {\scriptstyle pr_1}\nwarrow & \uparrow{\scriptstyle d} & \nearrow{\scriptstyle \mathfrak{h}} & \\ & & Y \times Y & & \end{array}$$

commutative and $\mathfrak{h}$ is a zero morphism. Now suppose that $\mathfrak{f}, \mathfrak{g} \in \mathcal{E}$ and $pr_1 \in \mathcal{M}$ are given such that $\rangle pr_1\mathfrak{f} \langle \subseteq \rangle \mathfrak{g} \langle$. Thus $pr_1\mathfrak{f} = v\mathfrak{g}$. So we have

$$\begin{array}{ccc} X & \xrightarrow{\mathfrak{f}} & Y \times Y \\ {\scriptstyle \mathfrak{g}}\downarrow & /// \; \overset{d}{\nearrow} & \downarrow{\scriptstyle pr_1} \\ Z \times Z & \xrightarrow[v]{} & Y \end{array}$$

where d is the unique morphism making the diagram

$$\begin{array}{ccccc} Y & \xleftarrow{pr_1} & Y \times Y & \xrightarrow{pr_2} & Y \\ & {\scriptstyle v}\nwarrow & \uparrow{\scriptstyle d} & \nearrow{\scriptstyle v} & \\ & & Z \times Z. & & \end{array}$$

commutes. Since $pr_1\mathfrak{f} = v\mathfrak{g}$ we have $f = v\mathfrak{g}$. Also $pr_1 d\mathfrak{g} = v\mathfrak{g} = f$ and $pr_2 d\mathfrak{g} = v\mathfrak{g} = f$, hence $d\mathfrak{g} = \mathfrak{f}$. Since $\mathrm{Iso}\,\mathcal{X} \nsubseteq \mathcal{M}$, this factorization system is not weak.

Example 4.12. In the category ***Grp***, let $\mathcal{E}$ and $\mathcal{M}$ be the following classes of morphisms:

$$\mathcal{E} = \{\, G \xrightarrow{\mathfrak{f}} G \times H \mid G \xrightarrow{f} H \text{ is a map and } \mathfrak{f} = \langle e, f\rangle, \text{ where } e \text{ is the zero map}\}$$
$$\mathcal{M} = \{\, A \times B \xrightarrow{pr_2} B \mid pr_2 \text{ is the second factor projection}\}.$$

For an arbitrary morphism $f : G \longrightarrow H$, we have $f = pr_2\mathfrak{f}$. Suppose that $\mathfrak{g} \in \mathcal{E}$ and $pr_2, pr'_2 \in \mathcal{M}$ are given such that $\langle pr_2\mathfrak{g}\rangle \subseteq \langle pr'_2\rangle$, thus $pr_2\mathfrak{g} = pr'_2 u$. So the map $d = \langle e, pr_2\rangle$ is a diagonal for the following diagram.

$$\begin{array}{ccc} G & \xrightarrow{u} & H \times K \\ {\scriptstyle \mathfrak{g}}\downarrow & & \downarrow{\scriptstyle pr'_2} \\ G \times K & \xrightarrow[pr_2]{} & K. \end{array}$$

So the condition (SW1) holds.

Now let $\mathfrak{f}, \mathfrak{g} \in \mathcal{E}$ and $pr_2 \in \mathcal{M}$ such that $G \xrightarrow{f} K$, $g : G \longrightarrow H$ and $\rangle pr_2\mathfrak{f}\langle \subseteq \rangle\mathfrak{g}\langle$ be given. Thus $pr_2\mathfrak{f} = v\mathfrak{g}$. So the map $d = \langle pr_1, v\rangle$ is a diagonal for the diagram

$$\begin{array}{ccc} G & \xrightarrow{\mathfrak{f}} & G \times K \\ {\scriptstyle \mathfrak{g}}\downarrow & & \downarrow{\scriptstyle pr_2} \\ G \times H & \xrightarrow[v]{} & K \end{array}$$

So the condition (SW2) holds. Therefore, $(\mathcal{E}, \mathcal{M})$ is a semi weak factorization system. Since $\mathrm{Iso}\,\mathcal{X} \nsubseteq \mathcal{M}$, this factorization system is not weak.

Remark 4.13. Note that examples 4.1 to 4.4 satisfy the conditions of Theorem 3.7.

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A. Ilaghi-Hosseini, *Department of Pure Mathematics, Shahid Bahonar University of Kerman, Kerman, Iran.*

Email: a.ilaghi@math.uk.ac.ir

S.Sh. Mousavi, *Department of Pure Mathematics, Shahid Bahonar University of Kerman, Kerman, Iran.*

Email: smousavi@uk.ac.ir

S.N. Hosseini, *Department of Pure Mathematics, Shahid Bahonar University of Kerman, Kerman, Iran.*

Email: nhoseini@uk.ac.ir