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Intersection graphs associated with semigroup acts

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Abstract. The intersection graph $\mathbb{Int}(A)$ of an S-act A over a semigroup S is an undirected simple graph whose vertices are non-trivial subacts of A, and two distinct vertices are adjacent if and only if they have a non-empty intersection. In this paper, we study some graph-theoretic properties of $\mathbb{Int}(A)$ in connection to some algebraic properties of A. It is proved that the finiteness of each of the clique number, the chromatic number, and the degree of some or all vertices in $\mathbb{Int}(A)$ is equivalent to the finiteness of the number of subacts of A. Finally, we determine the clique number of the graphs of certain classes of S-acts.

1 Introduction and Preliminaries

In recent decades, assigning graphs to algebraic structures has opened a new direction to study algebraic properties via graph-theoretic properties and vice versa. Several classes of graphs associated with algebraic structures have been extensively investigated by many authors in the literature (for

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example, [1–4, 6–8, 10, 12, 19, 21]). One such useful graph is the intersection graph which is important in both theoretical as well as in applications. For an overview of the theory of intersection graphs and important special classes of them, see [14]. The idea of studying algebraic properties of algebraic structures via their intersection graphs was initiated by Bosák [5] where the intersection graph of proper subsemigroups of a semigroup was considered. The intersection graph of some classes of ordered semigroups were briefly studied in [16, 17]. Some papers devoted to intersection graphs derived from other structures such as groups, rings, modules, and lattices have appeared during the years. Inspired by these studies, Rasouli and Tehranian [18] extended the idea to acts over semigroups and verified some elementary properties of the intersection graph $\mathbb{I}\mathrm{nt}(A)$ of non-trivial subacts of an S-act A over a semigroup S. Here we investigate more aspects of this graph and obtain more results. We try to relate the algebraic properties of an S-act to the graph-theoretic properties of its intersection graph.

Throughout S stands for a semigroup unless otherwise stated. Let A be an S-act. Recall from [18] that the *intersection graph* of A, denoted by $\mathbb{I}\mathrm{nt}(A)$, is an undirected simple graph whose vertices are non-trivial subacts of A and two distinct vertices B and C are adjacent if and only if $B \cap C \neq \emptyset$. Here we first investigate some properties of the graph $\mathbb{I}\mathrm{nt}(A)$ for an S-act A. Hereby, the interplay between some algebraic properties of an S-act A and graph-theoretic properties of $\mathbb{I}\mathrm{nt}(A)$ is considered. We study some graph-theoretic characters such as clique, chromatic, domination, and the independence numbers in $\mathbb{I}\mathrm{nt}(A)$, and prove the equivalences of the finiteness of the clique number, chromatic number, degree of some (or all) vertices, and the degree of $\mathbb{I}\mathrm{nt}(A)$, that is, the number of non-trivial subacts of A.

It is known that deciding whether a graph is weakly perfect is an NP-complete problem. A class of weakly perfect intersection graphs of ideals of a finite ring can be found in [15]. This result was generalized in [9], where Corollary 4.3 shows the intersection graph of submodules of any finite R-module (where R is any ring) is weakly perfect. In fact, the intersection graph of an intersection-closed family of non-empty subsets of a set is weakly perfect if it has finite clique number. As a consequence, the intersection graph of an S-act A with finite clique number is weakly perfect. This motivates us to determine the clique number of such graphs for some classes of S-acts. Regarding to the fact that each semigroup S can be viewed as an S-act over itself, it is worth noting that all results obtained here are also valid for S and the intersection graph $\mathbb{Int}(S)$ of non-trivial left ideals of S.

Let us give a brief account of some definitions about S-acts and graphs needed in the sequel.

Let S be a semigroup. A non-empty set A is said to be a (*left*) *S-act* if there is a mapping $\lambda : S \times A \to A$, denoting $\lambda(s, a)$ by sa, satisfying $(st)a = s(ta)$ and, if S is a monoid with 1, $1a = a$, for all $a \in A$ and $s, t \in S$. An element $\theta \in A$ is said to be a *fixed element* if $s\theta = \theta$ for all $s \in S$. A non-empty subset B of A is called a *subact* of A if it is closed under the action, that is, $sb \in B$, for every $s \in S$ and $b \in B$. By a *non-trivial* subact of an S-act A we mean a (non-empty) proper subact of A. The set of all non-trivial subacts of A is denoted by $\mathrm{Sub}(A)$. Clearly, S is an S-act with its operation as the action and so subacts of S are exactly the *left ideals* of S, that is, the non-empty subsets I of S satisfying $SI \subseteq I$. An element $z \in S$ is called a *left zero element* if $zs = z$ for all $s \in S$. If each element of S is a left zero element, then we say that S is a *left zero semigroup*. A non-empty S-act is said to be *simple* if it has no non-trivial subact. A non-trivial subact M of an S-act A is called a *minimal* subact of A if it properly contains no subact of A. We denote the set of all minimal subacts of A by $\mathrm{Min}(A)$. The *socle* of an S-act A, written as $\mathrm{Soc}(A)$, is the union of all minimal subacts of A. A *maximal* subact of A is a non-trivial subact M for which there is no subact of A properly contained between M and A. The *coproduct* of a family $\{A_i \mid i \in I\}$ of S-acts, denoted by $\coprod_{i\in I} A_i$, is their disjoint union $\bigcup_{i\in I}(A_i \times \{i\})$ with the action $s(a, i) = (sa, i)$ for every $s \in S$ and $a \in A_i, i \in I$. For more information about S-acts and related notions, the reader is referred to [13].

Let G be a graph with a vertex set $V(G)$. For distinct elements x and y of $V(G)$, an *x,y-path* (or $x - y$) is a path with starting vertex x and ending vertex y, and the length of the shortest x, y-path is denoted by $d(x, y)$. If G does not have such a path, then $d(x, y)=\infty$. By the *order* of G, denoted by $|G|$, we mean the number of vertices of G. The *diameter* of G, $diam(G)$, is the supremum of the set $\{d(x, y) : x, y \in V(G), x \neq y\}$. The number of vertices which are adjacent to x is called the *degree* of x and is denoted by $\deg(x)$. The *girth* of a graph is the length of its shortest cycle. A graph with no cycle has infinite girth. A *complete graph* with n vertices, denoted by K_n, is a graph in which every pair of distinct vertices are adjacent. For a graph G let $\chi(G)$ denote the *chromatic number* of G, that is, the minimum number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. A *clique* of G is a complete subgraph of G and the number of vertices in the largest clique of G, denoted by $\omega(G)$, is called the *clique number* of G. A graph G is called *weakly perfect* if $\chi(G) = \omega(G)$. For undefined terms and concepts, one may consult [20].

2 Some properties of the graph $\mathbb{Int}(A)$

In this section, we proceed with the study of some facts about the intersection graphs of S-acts.

It is of interest to know whether a graph is an intersection graph of an S-act. This property holds for every complete graph (see [18, Proposition 2.2]). Here the two classes of bipartite and wheel graphs which are intersection graphs of some S-acts are fully characterized.

A *bipartite graph* is a graph whose vertices can be partitioned into two sets in such a way that no two vertices within the same set are adjacent. Equivalently, a bipartite graph is a graph that contains no odd-length cycle. A *wheel graph* of order $n \geq 4$, denoted by W_n, is a graph formed by connecting a single vertex to all vertices of a cycle.

Theorem 2.1. *The following assertions hold*:

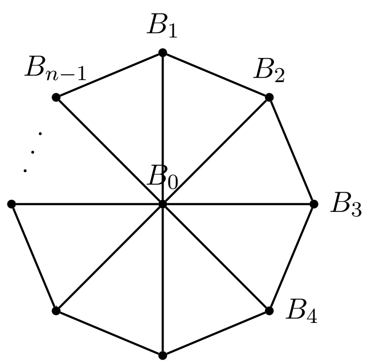
(i) *A bipartite graph G is the intersection graph of an S-act if and only if G is one of the graphs K_1, K_2, K_2^c and $K_{1,2}$.*

(ii) *The wheel graph W_n is the intersection graph of an S-act if and only if $n = 4$.*

Proof. (i) Suppose that $G = \mathbb{Int}(A)$ is bipartite where A is an S-act and $|G| > 3$. So A contains at least four non-trivial subacts. Let also $V_1 = \{B_1, B_2, \ldots, B_n\}$ and $V_2 = \{C_1, C_2, \ldots, C_m\}$ be two (non-empty) disjoint partitions of vertices of $\mathbb{Int}(A)$. Using [18, Theorem 3.1] and with no loss of generality, one can assume that B_1, C_1 are adjacent. We claim that $m, n \leq 2$. Let $C_1, C_2, C_3 \in V_2$. If $C_1 \cup C_2 = A$, then $C_3 \subseteq C_1 \cup C_2$ which follows that C_3 is adjacent to C_1 or C_2, which is a contradiction. Then $C_1 \cup C_2$ is a vertex of $\mathbb{Int}(A)$. If $C_1 \cup C_2 \in V_2$, then we have the path $C_1 - C_1 \cup$

$C_2 - C_2$ and if $C_1 \cup C_2 = B_i$ where $i \neq 1$, then B_1 is adjacent to B_i contradicting the hypothesis. Therefore, $C_1 \cup C_2 = B_1$. In the same way, $C_1 \cup C_3 = B_1$. Then we get $C_2 = C_3$, a contradiction. Hence, $|V_2| \leq 2$. By the same argument, $|V_1| \leq 2$. If $|V_1| = |V_2| = 2$, then, using [18, Theorems 3.1, 4.1], girth$(\mathbb{Int}(A)) = 3$ which contradicts the bipartitivity of $\mathbb{Int}(A)$. Consequently, $|G| = |\mathbb{Int}(A)| = |V_1| + |V_2| \leq 3$. Since K_2^c is the only disconnected intersection graph of an S-act by [18, Theorem 3.1], the assertion holds. The converse follows from [18, Proposition 2.2, Example 2.2, Theorems 3.1, 3.2].

(ii) Let $n \geq 5$ and suppose that there exists an S-act A with non-trivial subacts $B_0, B_1, B_2, \ldots, B_{n-1}$ such that the intersection graph $\mathbb{Int}(A)$ is the following wheel graph W_n:



Since $B_1 \cap B_2 \neq \emptyset$, $B_1 \cap B_2 = B_i$ for some $i \in \{0, 1, 2\}$. If $B_1 \cap B_2 = B_1$, then $B_{n-1} \cap B_2 \neq \emptyset$. If $B_1 \cap B_2 = B_2$, then $B_3 \cap B_1 \neq \emptyset$. If $B_1 \cap B_2 = B_0$, then $B_3 \cap B_1 \neq \emptyset$. In each case, we get a contradiction. The converse follows from [18, Proposition 2.2]. □

Theorem 2.1(ii) answers the question posed in [18] about the existence of a connected graph with diameter 2 and girth 3 which is the intersection graph of no S-act. In fact, all wheel graphs with at least five vertices satisfy the mentioned condition.

In view of Theorem 2.1(i) and [18, Theorem 4.1(ii)], the following is obtained.

Corollary 2.2. *For an S-act A, if $|\mathbb{Int}(A)| \geq 4$, then $\mathbb{Int}(A)$ contains K_3.*

An S-act A over a monoid S is called *free* if it has a *basis* X, that is, each element $a \in A$ is uniquely represented as $a = sx$ for some $s \in S$ and $x \in X$. In this case, $A \cong \coprod_{x\in X} S$. Moreover, A is isomorphic to the S-act $S \times X$ with the action given by $s(t,x) = (st,x)$ for all $s,t \in S, x \in X$ (see [13]).

If A and B are isomorphic S-acts, then their intersection graphs are clearly isomorphic; and there is an example which shows that this implication is strict (see [18, Example 2.3]). Here some conditions are posed to fill the gap. To this aim, first note the following:

Lemma 2.3. *Let A be a free S-act with a basis X where S is a group. Then $\mathbb{Int}(A) \cong \mathbb{Int}(X)$, in which X is considered as an S-act with trivial action.*

Proof. Using the assumption, A is isomorphic to the S-act $S \times X$. Since S is a group, non-trivial subacts of A (if exist) are of the forms $S \times Y$ where $Y \subset X$. Consider the set X as an S-act with trivial action. We claim that the graphs $\mathbb{Int}(A)$ and $\mathbb{Int}(X)$ are isomorphic. For this, define the map $f : \mathbb{Int}(A) \to \mathbb{Int}(X)$ by $f(S \times Y) = Y$ for any non-empty $Y \subset X$. Now it is straightforward to see that f is a graph isomorphism. $\square$

Theorem 2.4. *Let A and B be free S-acts and $\mathbb{Int}(A) \cong \mathbb{Int}(B)$. Then $A \cong B$ under each of the conditions*

(i) *S is a group,*

(ii) *S has only finitely many left ideals, and A and B have finite bases.*

Proof. (i) Assume that X and Y are bases of free S-acts A and B, respectively. Using Lemma 2.3, $\mathbb{Int}(A) \cong \mathbb{Int}(X)$ and $\mathbb{Int}(B) \cong \mathbb{Int}(Y)$, where X and Y are considered as S-acts with trivial actions. It follows from the assumption that $\mathbb{Int}(X) \cong \mathbb{Int}(Y)$ and then $2^{|X|} - 2 = |\mathrm{Sub}(X)| = |\mathrm{Sub}(Y)| = 2^{|Y|} - 2$. This implies that $|X| = |Y|$ and hence $A \cong B$.

(ii) This is trivial. $\square$

The next result presents some necessary and sufficient conditions for the graph $\mathbb{Int}(A)$ to be complete.

A graph G is said to be *r-regular* for some non-negative integer r if the degree of each vertex of G is r.

Proposition 2.5. *Let A be an S-act.*

(i) *If S contains a left zero element z, then $\mathbb{Int}(A)$ is complete if and only if $za = za'$ for all $a, a' \in A$.*

(ii) *Let* $3 \leq |\mathrm{Sub}(A)| < \infty$. *Then the following are equivalent:*
(1) $\mathbb{Int}(A)$ *is complete.*
(2) $\mathbb{Int}(A)$ *is* r*-regular for some* $r \in \mathbb{N}$.
(3) $|\mathrm{Min}(A)| = 1$.

Proof. (i) Let $\mathbb{Int}(A)$ be a complete graph and $a, a' \in A$. Consider the two subacts $B = Sa$ and $B' = Sa'$ of A. If $B, B' \neq A$, then $B \cap B' \neq \emptyset$ whence $sa = s'a'$ for some $s, s' \in S$. This gives that $za' = (zs')a' = z(s'a') = z(sa) = (zs)a = za$. If $B = A$ or $B' = A$, then the assertion clearly holds. For the converse, it suffices to note that for all non-trivial subacts B, B' of A, $zb = zb' \in B \cap B'$ for all $b \in B, b' \in B'$.

(ii) We need only to show the non-trivial direction (2) $\Rightarrow$ (1): If $\mathbb{Int}(A)$ is not complete, then $|\mathrm{Min}(A)| > 1$. Let M_1, M_2 be minimal subacts of A. Using the hypothesis and [18, Theorem 3.1], $\mathbb{Int}(A)$ is connected and hence $d(M_1, M_2) = 2$ by [18, Theorem 4.1(i)]. Thus there exists a non-trivial subact B of A such that $M_1 - B - M_2$ is a path between M_1 and M_2. It is clear that each vertex of $\mathbb{Int}(A)$ which is adjacent to M_1 and not equal to B is also adjacent to B. This gives that $\deg(B) > \deg(M_1)$, which is a contradiction. □

In the rest of this section, we restrict our attention to verify the existence of a cut vertex and a cut edge in $\mathbb{Int}(A)$.

A vertex v in a graph is a *cut vertex* if the removal of v and all edges with v as an end-point from the graph increases the number of components. A *cut edge* of a graph is an edge whose deletion (the end-points stay in place) from the graph increases the number of components.

Theorem 2.6. *Let every subact of an* S*-act* A *contain a minimal subact. Then the graph* $\mathbb{Int}(A)$ *has a cut vertex if and only if* $\mathrm{Soc}(A)$ *is the union of two minimal subacts and is a maximal subact of* A. *Moreover, in this case,* $\mathrm{Soc}(A)$ *is the unique cut vertex of* $\mathbb{Int}(A)$.

Proof. Assume that B is a cut vertex in $\mathbb{Int}(A)$. Then there exist two non-trivial subacts B_1, B_2 of A such that there is a path between B_1 and B_2 in $\mathbb{Int}(A)$ but no path between them in $\mathbb{Int}(A) - \{B\}$. Note that $\mathbb{Int}(A)$ must be connected because, otherwise, it has no cut vertex by [18, Theorem 3.1]. Then, using [18, Theorem 4.1(i)], $d(B_1, B_2) = 2$ in $\mathbb{Int}(A)$ and so $B_1 - B - B_2$ is a path from B_1 to B_2. Since $B_1 \cap B$ and $B_2 \cap B$ are disjoint non-trivial subacts of A, it follows from the assumption that there exist two (distinct) minimal subacts M_1, M_2 of A such that $M_1 \subseteq B_1 \cap B$ and $M_2 \subseteq B_2 \cap B$. We claim that $B = M_1 \cup M_2$. It is clear that $M_1 \cup M_2 \subseteq B$. If $M_1 \cup M_2 \subset B$, then the path $B_1 - M_1 \cup M_2 - B_2$ is a path from B_1 to B_2 in $\mathbb{I}\mathrm{nt}(A) - \{B\}$, which is a contradiction. If $B \neq \mathrm{Soc}(A)$, then there exists a minimal subact M_3 of A such that $B_1 - M_1 \cup M_3 - M_2 \cup M_3 - B_2$ is a path from B_1 to B_2 in $\mathbb{I}\mathrm{nt}(A) - \{B\}$, which is a contradiction. Hence, $\mathrm{Soc}(A) = M_1 \cup M_2$ is a cut vertex of $\mathbb{I}\mathrm{nt}(A)$. If there exists a non-trivial subact C of A such that $\mathrm{Soc}(A) \subset C$, then $B_1 - C - B_2$ is a path from B_1 to B_2 in $\mathbb{I}\mathrm{nt}(A) - \{B\}$, which is a contradiction. This means that $\mathrm{Soc}(A)$ is a maximal subact of A.

For the converse, let $\mathrm{Soc}(A) = M_1 \cup M_2$ where M_1, M_2 are minimal subacts of A and $\mathrm{Soc}(A)$ is a maximal subact of A. We claim that $\mathrm{Soc}(A)$ is a cut vertex of $\mathbb{I}\mathrm{nt}(A)$. First note that $M_1 - \mathrm{Soc}(A) - M_2$ is a path between M_1 and M_2 in $\mathbb{I}\mathrm{nt}(A)$. Suppose that there exists a path $M_1 - B_1 - B_2 - \cdots - B_n - M_2$ in $\mathbb{I}\mathrm{nt}(A) - \{\mathrm{Soc}(A)\}$. This clearly gives that $M_1 \subseteq B_1$ and $M_2 \subseteq B_n$. We claim that there exists $i \in \{1, \ldots, n\}$ such that $\mathrm{Soc}(A) \subseteq B_i$, whence $\mathrm{Soc}(A) = B_i$, by the maximality of $\mathrm{Soc}(A)$, which is a contradiction. To do so, note that if $M_1 \subseteq B_n$, then $\mathrm{Soc}(A) = M_1 \cup M_2 \subseteq B_n$ and we are done. Otherwise, $M_1 \nsubseteq B_n$. Let $1 \leq k < n$ be the greatest positive integer for which $M_1 \subseteq B_k$. We show that $\mathrm{Soc}(A) \subseteq B_k$. We have $B_k \cap B_{k+1} \neq \emptyset$. The choice of k implies that $M_1 \nsubseteq B_k \cap B_{k+1}$. Then, using the assumption, $M_2 \subseteq B_k \cap B_{k+1}$ and, hence, $\mathrm{Soc}(A) = M_1 \cup M_2 \subseteq B_k$, as claimed. This completes the proof. □

Definition 2.7. Let A be an S-act and B, C be subacts of A. We say that C *covers* B (or C *is a cover for* B), denoted by $B \sqsubset C$, if $B \subset C$ and no element in $\mathrm{Sub}(A)$ lies strictly between B and C, that is, $B \subseteq D \subseteq C$ implies that $D = B$ or $D = C$.

Lemma 2.8. *Let an edge e with end-points B_1 and B_2 be a cut edge in $\mathbb{I}\mathrm{nt}(A)$. Then, without loss of generality, B_1 is a minimal subact of A and B_2 is a maximal subact of A as well as the unique cover for B_1.*

Proof. Since e is a cut edge, there is no path between B_1 and B_2 other than e in $\mathbb{I}\mathrm{nt}(A)$. As $B_1 \cap B_2 \neq \emptyset$, if B_1, B_2 and $B_1 \cap B_2$ are all distinct, then $B_1 - B_1 \cap B_2 - B_2$ is a path between B_1 and B_2, which is a contradiction. Hence, we may assume that $B_1 \subset B_2$. If there is a non-trivial subact B of A such that $B \subset B_1$ or $B_2 \subset B$ or $B_1 \subset B \subset B_2$, then we have the path $B_1 - B - B_2$, which is a contradiction. So the subacts B_1 and B_2 are minimal and maximal, respectively, and $B_1 \sqsubset B_2$. If C is another cover for B_1, then we have the path $B_1 - C - B_2$, which is a contradiction. $\square$

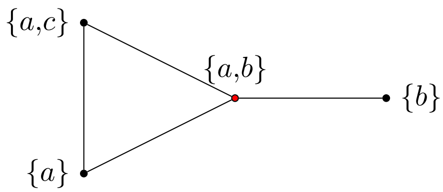
Theorem 2.9. *Let A be an S-act. If $\mathbb{Int}(A)$ has a cut edge, then $|\mathbb{Int}(A)| \leq 4$.*

Proof. Assume on the contrary that A has at least five non-trivial subacts. Let us denote the set $\mathrm{Sub}(A)$ by $\{B_1, B_2, B_3, \ldots\}$. Let also e be a cut edge of $\mathbb{Int}(A)$ with end-points, say, B_1 and B_2. In light of Lemma 2.8, with no loss of generality, B_1 is a minimal subact of A which is properly contained in B_2. If B_1 intersects B_i for some $i > 2$, then it follows from the minimality of B_1 that $B_1 \subseteq B_i$ and so we have the path $B_1 - B_i - B_2$, which is a contradiction. Thus $B_1 \cap B_i = \emptyset$ for all $i > 2$. If $B_1 \cup B_3 \in \{B_4, B_5, \ldots\}$, then there is the path $B_1 - B_1 \cup B_3 - B_2$ and if $B_1 \cup B_4 \in \{B_3, B_5, \ldots\}$, then there is the path $B_1 - B_1 \cup B_4 - B_2$. Each case yields a contradiction. This gives that $B_1 \cup B_3, B_1 \cup B_4 \in \{B_2, A\}$. Since $B_1 \cup B_3 \neq B_1 \cup B_4$, without loss of generality, $B_1 \cup B_3 = B_2$ and $B_1 \cup B_4 = A$. Thus, $B_1 \cup B_5 \notin \{A, B_1, B_2\}$, so there is a path $B_1 - B_1 \cup B_5 - B_2$, which is a contradiction. $\square$

Corollary 2.10. *Let A be an S-act and $\mathbb{Int}(A)$ has a vertex of degree 1. Then $|\mathbb{Int}(A)| \leq 4$.*

Proof. Let B be a vertex of degree 1 and e be the only edge adjacent to B in $\mathbb{Int}(A)$. Clearly, e is a cut edge in $\mathbb{Int}(A)$, which implies that $|\mathbb{Int}(A)| \leq 4$, by Theorem 2.9. $\square$

Example 2.11. (i) Consider the monoid $S = \{1, s\}$, where s is an idempotent element, and $A = \{a, b, c\}$ with the action defined by $1c = c, sc = a$, and a, b are fixed elements. Then all non-trivial subacts of A are the sets $\{a\}, \{b\}, \{a, b\}$ and $\{a, c\}$. Since $\mathrm{Soc}(A) = \{a, b\}$ is the union of two minimal subacts of A and is a maximal subact of A, using Theorem 2.6, $\{a, b\}$ is the only cut vertex of the graph $\mathbb{Int}(A)$. Moreover, the edge $\{a, b\} - \{b\}$ is the only cut edge of $\mathbb{Int}(A)$, as follows:



(ii) If A is an S-act with trivial action, then $\mathrm{Soc}(A) = A$. This implies that the graph $\mathbb{Int}(A)$ has no cut vertex, by Theorem 2.6.

3 Some finiteness conditions

In this section, we study finiteness conditions of some parameters of intersection graphs of S-acts such as clique number and chromatic number.

We say that an S-act A is *Artinian* (*Noetherian*) if every descending (ascending) chain of subacts of A terminates. It is clear that every subact of an Artinian S-act contains a minimal subact.

Remark 3.1. Consider any S-act A with infinitely many pairwise disjoint non-trivial subacts, say $B_1, B_2, B_3, \ldots$. Then A is neither Noetherian nor Artinian, whence $\omega(\mathbb{Int}(A)) = \infty$. Indeed, the infinite strict ascending chain $B_1 \subset B_1 \cup B_2 \subset B_1 \cup B_2 \cup B_3 \subset \cdots$ of subacts of A gives an infinite clique in $\mathbb{Int}(A)$. Moreover, with no loss of generality, one can assume that the subact $\bigcup_{i=1}^{\infty} B_i$ is non-trivial. In this case, we have the infinite strict descending chain $\bigcup_{i=1}^{\infty} B_i \supset \bigcup_{i=2}^{\infty} B_i \supset \bigcup_{i=3}^{\infty} B_i \supset \cdots$ of subacts of A which gives another infinite clique in $\mathbb{Int}(A)$ with no adjacent vertex with the previous one. As a consequence, if A is a Noetherian or an Artinian S-act, then the set $\mathrm{Min}(A)$ is finite. For instance, let $\{A_i\}_{i=1}^{\infty}$ be a family of S-acts and $A = \coprod_{i=1}^{\infty} A_i$. Since $A_1, A_2, A_3, \ldots$ are pairwise disjoint non-trivial subacts of A, $\omega(\mathbb{Int}(A)) = \infty$.

In the following, a main result concerning finiteness of clique and chromatic numbers of the graph $\mathbb{Int}(A)$ is presented.

Theorem 3.2. *Let A be an S-act. Then the following are equivalent*:

(i) $\deg(B) < \infty$ *for some vertex B in* $\mathbb{Int}(A)$.

(ii) $\deg(B) < \infty$ *for each vertex B in* $\mathbb{Int}(A)$.

(iii) $\omega(\mathbb{Int}(A)) < \infty$.

(iv) $\chi(\mathbb{Int}(A)) < \infty$.

(v) $|\mathbb{Int}(A)| < \infty$, *that is, the number of subacts of* A *is finite.*

Proof. The implications (v) $\Rightarrow$ (iv) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (i) are trivial.

(i) $\Rightarrow$ (v) Assume that $\deg(B) < \infty$ for some vertex B in $\mathbb{Int}(A)$. Suppose, on the contrary, that $|\mathbb{Int}(A)| = \infty$. Since $\deg(B) < \infty$, there exist infinitely many (pairwise distinct) $B_i \in \mathrm{Sub}(A), i \in I$, for which $B_i \cap B = \emptyset$. Thus, $B_i \cup B \neq B_j \cup B$ for all $i \neq j$. Hence, $\{B_i \cup B\}_{i \in I}$ contains infinitely many vertices of $\mathbb{Int}(A)$ that are adjacent to B, which is a contradiction.

(iii) $\Rightarrow$ (ii) Suppose that there exists a vertex B and an infinite set $W = \{B_i : i \in I\}$ such that B_i is adjacent to B for all $i \in I$. By the well-known *Infinite Ramsey's Theorem*, the subgraph of $\mathbb{Int}(A)$ induced by W contains either an infinite clique or an infinite set of pairwise disjoint subacts. Both cases yield an infinite clique in $\mathbb{Int}(A)$ (see Remark 3.1), so if (iii) holds then (ii) holds. □

As a direct consequence of Theorem 3.2, a useful result in semigroup theory is obtained below.

Corollary 3.3. *For a semigroup* S, *we have the following equivalent assertions*:

(i) *There is a non-trivial left ideal of* S *intersecting only finitely many left ideals.*

(ii) *Any non-trivial ideal of* S *intersects only finitely many left ideals.*

(iii) $\mathbb{Int}(S)$ *is colored by finitely many colors.*

(iv) S *has finitely many left ideals.*

The following example shows that the clique number of the graph $\mathbb{Int}(A)$ is not necessarily finite.

Example 3.4. (i) Take the monoid $S = (\mathbb{N}^\infty, \min)$ where $n < \infty$ for all $n \in \mathbb{N}$. The non-trivial ideals of S are exactly the principal ones $\downarrow k = \{x \in \mathbb{N}^\infty \mid x \leq k\}, k \in \mathbb{N}$, and its only non-principal ideal $\mathbb{N}$ (see [11, Remark 4]). Note that $\downarrow m \subset \downarrow n$ if and only if $m < n$ for every $m, n \in \mathbb{N}$. Therefore, the graph $\mathbb{Int}(A)$ is complete with $\omega(\mathbb{Int}(A)) = \infty$.

(ii) Consider the semigroup $S = (\mathbb{N}, +)$. It is easily seen that non-trivial ideals of S are exactly the sets $\mathbb{N}_k = \{k+1, k+2, \ldots\}$ where $k \in \mathbb{N}$; and $\mathbb{N}_m \subset \mathbb{N}_n$ if and only if $n < m$ for every $m, n \in \mathbb{N}$. Then $\mathbb{Int}(A)$ is complete with $\omega(\mathbb{Int}(A)) = \infty$.

Proposition 3.5. *The following statements are satisfied:*

(i) *Let* $\{A_i\}_{i=1}^n$ *be a family of* S*-acts and* $A = \coprod_{i=1}^n A_i$*. Then* $\chi(\mathbb{Int}(A)) < \infty$ *if and only if* $\chi(\mathbb{Int}(A_i)) < \infty$ *for all* $i \in \{1, \ldots, n\}$*.*

(ii) *For a free* S*-act* A *with finite basis,* $\chi(\mathbb{Int}(A)) < \infty$ *if and only if* $\chi(\mathbb{Int}(S)) < \infty$*.*

Proof. (i) It suffices to note that $|\mathrm{Sub}(A)| < \infty$ if and only if $|\mathrm{Sub}(A_i)| < \infty$ for all $i \in \{1, \ldots, n\}$. Now apply Theorem 3.2.

(ii) It follows from (i). □

We close this section with some results on the domination number and independence number of the graph $\mathbb{Int}(A)$. Let us give some definitions.

Let G be a graph. The (open) neighborhood $N(x)$ of a vertex $x \in V(G)$ is the set of vertices which are adjacent to x. For a subset T of vertices, we put $N(T) = \bigcup_{x \in T} N(x)$ and $N[T] = N(T) \cup T$. If $N[T] = V(G)$, then T is said to be a *dominating set*. It is clear that every vertex not in a dominating set T is adjacent to a vertex in T. The *domination number* of G, $\gamma(G)$, is the minimum cardinality of a dominating set of G. An *independent set* in a graph is a set of pairwise non-adjacent vertices. The *independence number* of G, written by $\alpha(G)$, is the maximum size of independent sets.

Theorem 3.6. *Let* A *be an Artinian* S*-act. Then the following assertions hold:*

(i) $\mathrm{Min}(A)$ *is an independent as well as a dominating set.*

(ii) $\alpha(\mathbb{Int}(A)) = |\mathrm{Min}(A)|$.

(iii) $\gamma(\mathbb{Int}(A)) \leq \alpha(\mathbb{Int}(A))$.

(iv) $\gamma(\mathbb{Int}(A)) = 1$ *or* 2.

Proof. (i) It is clear.

(ii) Using (i), $\alpha(\mathbb{Int}(A)) \geq |\mathrm{Min}(A)|$. In view of Remark 3.1, suppose that $|\mathrm{Min}(A)| = n$. If $\alpha(\mathbb{Int}(A)) = m > n$ and $W = \{C_1, C_2, \ldots, C_m\}$ is an independent set in $\mathbb{Int}(A)$ of size m, then it follows from the hypothesis that there are distinct subacts C_i, C_j in W such that they contain a same minimal subact. Thus $C_i \cap C_j \neq \emptyset$, which is a contradiction.

(iii) It follows from (i) and (ii).

(iv) If A has only one minimal subact, say M, then $\{M\}$ is a dominating set and so $\gamma(\mathbb{Int}(A)) = 1$. If $\mathrm{Min}(A) = \{M_i : i \in I\}$ with $|I| \geq 2$, then $\{\bigcup_{i \in I, i \neq j} M_i, M_j\}$ forms a dominating set in $\mathbb{Int}(A)$ and hence $\gamma(\mathbb{Int}(A)) \leq 2$. □

4 On clique number of $\mathbb{Int}(A)$

As we mentioned in Introduction, it follows from [9, Corollary 4.3] that the intersection graph of an S-act with finite clique number is weakly perfect. In this section, we find the clique number (equivalently, chromatic number) of such graphs for some classes of S-acts.

The intersection graph of an S-act A with countably infinitely many subacts is also weakly perfect. Indeed, using Theorem 3.2, $\chi(\mathbb{Int}(A)) = \omega(\mathbb{Int}(A)) = \aleph_0$.

It should be noted that the finiteness of $\omega(\mathbb{Int}(A))$ for an S-act A implies that A contains finitely many subacts (see Theorem 3.2) and then each non-trivial subact of A contains a minimal subact. This fact is implicitly used in the proof of the following result.

Theorem 4.1. *Let A be an S-act whose graph $\mathbb{Int}(A)$ has finite clique number. Then the following assertions hold*:

(i) *If $|\mathrm{Min}(A)| = 2$, then $\omega(\mathbb{Int}(A)) = m + 1$, where m is the maximum degree of minimal subacts of A.*

(ii) *If $|\mathrm{Min}(A)| = 3$ and $A = \mathrm{Soc}(A)$, then $\omega(\mathbb{Int}(A)) = 3$.*

(iii) *If S is a group and $|\mathbb{Int}(A)| = n$, then n is even and $\omega(\mathbb{Int}(A)) = \frac{1}{2}n$.*

(iv) *If the action of S on A is trivial and $|A| = n$, then $|\mathbb{Int}(A)| = 2^n - 2$ and $\omega(\mathbb{Int}(A)) = 2^{n-1} - 1$.*

Proof. (i) Let $\mathrm{Min}(A) = \{M_1, M_2\}$ and $m_i = \deg(M_i) + 1$ which is the number of non-trivial subacts of A containing $M_i, i \in \{1, 2\}$ and assume, with no loss of generality, that $m_1 \geq m_2$. We claim that $\omega(\mathbb{Int}(A)) = m_1$. First note that the set $\mathrm{Sub}(A)$ can be partitioned to the following (possibly empty) subsets:

$$W_1 := \{B \in \mathrm{Sub}(A) \mid \mathrm{Soc}(B) = M_1\},$$
$$W_2 := \{B \in \mathrm{Sub}(A) \mid \mathrm{Soc}(B) = M_2\},$$
$$W_3 := \{B \in \mathrm{Sub}(A) \mid \mathrm{Soc}(B) = M_1 \cup M_2\}.$$

Moreover, $|W_1| + |W_3| = m_1$ and $|W_2| + |W_3| = m_2$. The elements of W_1 induce a clique of order $|W_1|$ in $\mathbb{Int}(A)$ which are colored by $|W_1|$ colors. On the other hand, for any $B_1 \in W_1$ and $B_2 \in W_2$, $B_1 \cap B_2 = \emptyset$ and it follows from $m_1 \geq m_2$ that $|W_1| \geq |W_2|$. Therefore, the elements of W_2 forming a clique of order $|W_2|$ in $\mathbb{Int}(A)$ are colored by $|W_2|$ colors of the elements of W_1. This means that the elements of $W_1 \cup W_2$ are colored by exactly $|W_1|$ colors. Also the elements of W_3 are colored by $|W_3|$ colors different from that of W_1 because the elements of W_3 form a clique of order $|W_3|$ and $B_1 \cap B_3 \neq \emptyset$ for any $B_1 \in W_1$ and $B_3 \in W_3$. Furthermore, it is clear that the elements of $W_1 \cup W_3$ induce the largest clique in $\mathbb{Int}(A)$. Consequently, $\omega(\mathbb{Int}(A)) = \chi(\mathbb{Int}(A)) = |W_1| + |W_3| = m_1$.

(ii) Let $\mathrm{Min}(A) = \{M_1, M_2, M_3\}$ and $M_1 \cup M_2 \cup M_3 = A$. We show that M_1, M_2, M_3, $M_1 \cup M_2$, $M_1 \cup M_3$ and $M_2 \cup M_3$ are all of the non-trivial subacts of A. To this end, consider another non-trivial subact B of A and $M_1 \subset B$, say. This gives that either $B \cap M_2 \neq \emptyset$ or $B \cap M_3 \neq \emptyset$. So either $M_2 \subseteq B$ or $M_3 \subseteq B$. Suppose, without loss of generality, that $M_2 \subseteq B$ and $M_3 \nsubseteq B$. Then $B \cup M_3 = A$ and $B \cap M_3 = \emptyset$. This implies that $B = M_1 \cup M_2$, which is a contradiction. Hence, it is clear that $\omega(\mathbb{Int}(A)) = 3$.

(iii) Suppose that $|\mathbb{Int}(A)| = n$. Let $\mathrm{Min}(A) = \{M_1, M_2, \ldots, M_t\}$ and m_i be the number of non-trivial subacts of A which contains M_i for all $i \in \{1, \ldots, t\}$. First we show that n is even and $m_1 = m_2 = \cdots = m_t = \frac{1}{2}n$. For this, take $V_i := \{B \in \mathrm{Sub}(A) \mid M_i \subseteq B\}$ and $W_i := \mathrm{Sub}(A) \setminus V_i$ for every $i \in \{1, \ldots, t\}$. Being S a group implies that the map $f : V_i \to W_i$ given by $f(B) = A \setminus B$, for any $B \in V_i$, is a one to one correspondence so that $|W_i| = |V_i| = m_i, 1 \leq i \leq t$. It follows that $n = |\mathbb{Int}(A)| = |\mathrm{Sub}(A)| = |V_i| + |W_i| = m_i + m_i = 2m_i$, as required. We claim that $\chi(\mathbb{Int}(A)) = \omega(\mathbb{Int}(A)) = \frac{1}{2}n$. Take $i = 1$. The elements of V_1 form a clique of order m_1 in $\mathbb{Int}(A)$ and m_1 colors are needed for coloring the elements of V_1. Moreover, any $B \in W_1$ can be colored by the same color of its complement $A \setminus B$ in V_1. Therefore, $m_1 \leq \omega(\mathbb{Int}(A)) \leq \chi(\mathbb{Int}(A)) = m_1$ and hence $\omega(\mathbb{Int}(A)) = \chi(\mathbb{Int}(A)) = m_1 = \frac{1}{2}n$.

(iv) Let $A = \{a_1, a_2, \ldots, a_n\}$. Clearly, $|\mathbb{Int}(A)| = |\mathrm{Sub}(A)| = 2^n - 2$. Define $W_1 := \{B \in \mathrm{Sub}(A) \mid a_1 \in B\}$ and $W_2 := \mathrm{Sub}(A) \setminus W_1$. It is obvious that $|W_1| = |W_2| = 2^{n-1} - 1$. The elements of W_1 form a clique in $\mathbb{Int}(A)$ and so $2^{n-1} - 1$ colors are needed for coloring them. Furthermore, any $B \in W_2$ can be colored by the same color of its complement $A \setminus B$ in W_1. Thus $2^{n-1} - 1 \leq \omega(\mathbb{Int}(A)) \leq \chi(\mathbb{Int}(A)) = 2^{n-1} - 1$ and then the assertion holds. □

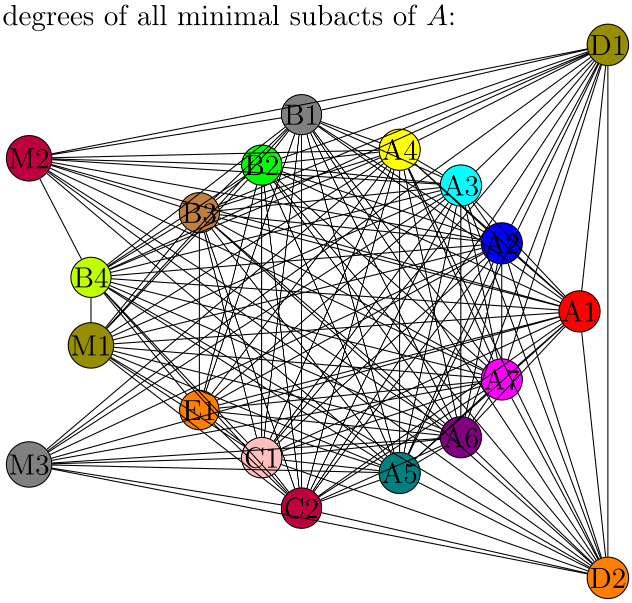
Open Problem 4.2. For every S-act A, if the graph $\mathbb{Int}(A)$ has finite clique number, is $\omega(\mathbb{Int}(A))$ equal to $m+1$, where m is the maximum degree of minimal subacts of A?

Theorem 4.1 and the following example give a positive answer to the above problem in some particular cases.

Example 4.3. Take the monoid $S = \{1, s, t\}$ where $S = \{s, t\}$ is a left zero semigroup. Consider the S-act $A = \{z_1, z_2, z_3, a, b, c\}$ with three fixed elements z_1, z_2, z_3, and $sa = ta = sb = z_1, tb = sc = z_2, tc = z_3$ (see [13, Example I.5.5(6)]). The non-trivial subacts of A are listed as

$A_1 = \{z_1, z_2, z_3\}$	$A_2 = \{z_1, z_2, z_3, a\}$	$A_3 = \{z_1, z_2, z_3, b\}$
$A_4 = \{z_1, z_2, z_3, c\}$	$A_5 = \{z_1, z_2, z_3, b, c\}$	$A_6 = \{z_1, z_2, z_3, a, c\}$
$A_7 = \{z_1, z_2, z_3, a, b\}$	$B_1 = \{z_1, z_2\}$	$B_2 = \{z_1, z_2, b\}$
$B_3 = \{z_1, z_2, a\}$	$B_4 = \{z_1, z_2, a, b\}$	$C_1 = \{z_1, z_3, a\}$
$C_2 = \{z_1, z_3\}$	$D_1 = \{z_2, z_3\}$	$D_2 = \{z_2, z_3, c\}$
$E_1 = \{z_1, a\}$	$M_1 = \{z_1\}$	$M_2 = \{z_2\}$
$M_3 = \{z_3\}$.		

The subacts M_1, M_2, and M_3 are all minimal subacts of A which are of the degrees 14, 13, and 11, respectively. Also $\mathbb{Int}(A)$ is weakly perfect and $\chi(\mathbb{Int}(A)) = \omega(\mathbb{Int}(A)) = 15 = \deg(M_1) + 1$ in which $\deg(M_1)$ is maximum between the degrees of all minimal subacts of A:



A *planar graph* is one which has a drawing in the plane without edge crossing. A well-known characterization of the planar graphs states that a graph is planar if and only if it contains no subgraph which is a subdivision of K_5 or $K_{3,3}$.

Proposition 4.4. *Let* $\mathbb{Int}(A)$ *be a planar graph for an* S*-act* A*. Then the following assertions hold*:

(i) A *is both Artinian and Noetherian.*

(ii) $1 \leq |\text{Min}(A)| \leq 3$.

(iii) *If* $|\text{Min}(A)| = 3$*, then* $\omega(\mathbb{Int}(A)) \in \{3, 4\}$.

Proof. (i) Any infinite increasing chain or decreasing chain of non-trivial subacts of A gives K_5 as a subgraph of $\mathbb{Int}(A)$, which contradicts the planarity of $\mathbb{Int}(A)$.

(ii) Suppose, on the contrary, that M_1, M_2, M_3, and M_4 are minimal subacts of A. Then the distinct non-trivial subacts $M_1, M_1 \cup M_2, M_1 \cup M_3, M_1 \cup M_4$ and $M_1 \cup M_2 \cup M_3$ form the subgraph K_5 of $\mathbb{Int}(A)$, which is a contradiction. Moreover, using (i), A contains at least one minimal subact. Then the assertion holds.

(iii) In view of Theorem 4.1(ii), it remains to consider the case $A \neq M_1 \cup M_2 \cup M_3$ in which M_1, M_2, and M_3 are three minimal subacts of A, noting that any non-trivial subact of A contains one of them by (i). We claim that A has only seven non-trivial subacts, which are $M_1, M_2, M_3, M_1 \cup M_2, M_1 \cup M_3, M_2 \cup M_3$, and $M_1 \cup M_2 \cup M_3$. Indeed, let B be another non-trivial subact of A. Without loss of generality, assume that $M_1 \subset B$. Then the subacts $M_1, M_1 \cup M_2, M_1 \cup M_3, M_1 \cup M_2 \cup M_3$, and B form K_5, which is a contradiction. Now it is easy to see that $\omega(\mathbb{Int}(A)) = 4$. □

Remark 4.5. If $\mathbb{Int}(A)$ is planar, where A is a free S-act over a monoid S, then S is a group or isomorphic to A. Indeed, $A = S \times X$, where X is a non-empty set. On the contrary, suppose that S is a non-group and non-isomorphic to A. Then there exist a non-trivial ideal I of S and distinct elements x_1, x_2 in X. Then we have the non-trivial subacts $B_i = S \times \{x_i\}, C_i = I \times \{x_i\}, i = 1, 2$ of A. Now it is easily seen that the non-trivial subacts $B_1, C_1, B_2 \cup C_1, B_1 \cup C_2$, and $C_1 \cup C_2$ form the subgraph K_5 of $\mathbb{Int}(A)$, which contradicts the planarity of $\mathbb{Int}(A)$.

Example 4.6. (i) The converse of Proposition 4.4(ii) is not true in general. For this, see Example 3.4(i) where $|\text{Min}(A)| = 1$ and $\mathbb{Int}(A)$ is not planar.

(ii) Let A be an S-act with trivial action. Then $|\mathrm{Min}(A)| = |A|$. Applying Theorem 3.6, we get $\alpha(\mathbb{Int}(A)) = |A|$. Further, if $|A| > 3$, then the graph $\mathbb{Int}(A)$ is not planar by Proposition 4.4(ii).

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