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(r, t)-injectivity in the category S-Act

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Dedicated to George A. Grätzer

Abstract. In this paper, we show that injectivity with respect to the class $\mathcal{D}$ of dense monomorphisms of an idempotent and weakly hereditary closure operator of an arbitrary category well-behaves. Indeed, if $\mathcal{M}$ is a subclass of monomorphisms, $\mathcal{M} \cap \mathcal{D}$-injectivity well-behaves. We also introduce the notion of (r, t)-injectivity in the category **S-Act**, where r and t are Hoehnke radicals, and discuss whether this kind of injectivity well-behaves.

1 introduction and preliminaries

Various generalizations of injectivity have been studied in various categories for their own and also tightly related notions such as purity, complete Boolean algebras (in the category of distributive lattices), essential monomorphisms, exponentiability, etc, see [2, 3, 12, 13, 18, 19, 21]. $\mathcal{M}$-injectivity, for which $\mathcal{M}$ is a subclass of morphisms, is one of these generalizations which has been captured the interest of many mathematicians in different fields, [3, 11, 12, 21]. Here we concentrate on another generalization of injectivity, that is, $(\mathcal{M}, \mathcal{E})$-injectivity, which is also studied in

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different branches of mathematics, see for example [20]. Indeed, given two arbitrary classes of morphisms $\mathcal{M}$ and $\mathcal{E}$ in a category $\mathcal{C}$, we say that an object Q is $(\mathcal{M}, \mathcal{E})$-*injective* if any $\mathcal{E}$-morphism $f : A \to Q$ can be extended through every $\mathcal{M}$-morphism $m : A \to B$. More explicitly, Q is $(\mathcal{M}, \mathcal{E})$-injective if for every $f : A \to Q$ in $\mathcal{E}$ and every $m : A \to B$ in $\mathcal{M}$, there exists a morphism $\bar{f} : B \to Q$ such that $\bar{f}m = f$. When $\mathcal{E}$ is the class of all morphisms the notions of $\mathcal{M}$-injectivity and $(\mathcal{M}, \mathcal{E})$-injectivity coincide.

Here, we focus on a special kind of $(\mathcal{M}, \mathcal{E})$-injectivity, that is, (r, t)-injectivity, in the category **S-Act**, where r and t are Hoehnke radicals. Specially, we show that for some Hoehnke radicals r and t, the notion of (r, t)-injectivity well-behaves.

Recall from [3] that the notion of $\mathcal{M}$-injectivity *well-behaves* if every object has an $\mathcal{M}$-injective hull $E_{\mathcal{M}}(A)$ which is both maximal $\mathcal{M}$-essential extension and minimal $\mathcal{M}$-injective extension of A, and every $\mathcal{M}$-injective object is $\mathcal{M}$-absolute retract and has no proper $\mathcal{M}$-essential extension.

In this paper, we first show that the notion of $\mathcal{M} \cap \mathcal{D}$-injectivity well-behaves in a category $\mathcal{C}$ when $\mathcal{M}$ is a subclass of monomorphisms and $\mathcal{D}$ is the class of dense monomorphisms of an idempotent and weakly hereditary closure operator of the category $\mathcal{C}$. Then, in Sections 3 and 4, we investigate a special type of $(\mathcal{M}, \mathcal{E})$-injectivity in the category **S-Act** which is related to Hoehnke radicals r and t, that is, (r, t)-injectivity. In fact (r, t)-injectivity in **S-Act** is the counterpart of the notion of (ρ, σ)-injectivity in which ρ and σ are radicals, in the category **R-Mod** of R-modules [4, 5, 16, 19]. Eventually, in Section 5, using the given criterion in Section 2 and employing t-subacts defined in Section 4, we show that for some Hoehnke radicals r and t, (r, t)-injectivity well-behaves.

Now let us briefly recall some necessary notions needed in this paper.

An S-*act* A over a monoid S is a set A together with an action $(s, a) \mapsto sa$, for $a \in A$, $s \in S$, subject to the rules $s(ta) = (st)a$ and $1a = a$, where 1 is the identity element of the monoid S, $a \in A$ and $s, t \in S$. We will work in the category of all S-acts and all homomorphisms $f : A \to B$, defined by $f(sa) = sf(a)$, for all $a \in A$ and $s \in S$. An element z of an S-act A is said to be a *zero* if $sz = z$, for all $s \in S$. Also, we say that an S-act A is *trivial* if $|A| \leq 1$.

An equivalence relation χ on an S-act A is called a *congruence* on A if $a\chi a'$ implies $(sa)\chi(sa')$, for all $s \in S$. We denote the set of all congruences on A by $\mathtt{Con}(A)$, which forms a lattice, see [6]. In the lattice $\mathtt{Con}(A)$ there is the smallest congruence, the diagonal relation $\Delta_A = \{(a,a)|a \in A\}$, and the largest congruence, the total relation $\nabla_A = \{(a,b)|\ a,b \in A\}$.

Every congruence $\chi \in \mathtt{Con}(A)$ determines a partition of A into χ-classes and a set Σ_χ of those χ-classes each of which is a nontrivial subact of A. Of course, Σ_χ may be empty. Throughout this paper, we use the general Rees congruence introduced in [22]; that is, in a *general Rees congruence* the classes are either subacts or consist of one element. Also, every set Σ of disjoint nontrivial subacts of an S-act A determines a general Rees congruence ρ_Σ given by

$$(a,b) \in \rho_\Sigma \Longleftrightarrow \begin{cases} a,b \in B & \text{for some } B \in \Sigma \\ a = b & \text{otherwise.} \end{cases}$$

We call ρ_Σ the *general Rees congruence generated by* Σ on A and A/ρ_Σ the *general Rees factor* of A over ρ_Σ (or for short, the general Rees factor). It is worth noting that in the case $\Sigma = \{B\}$ the general Rees congruence and Rees congruence coincide. In this case, we use the notations ρ_B and A/B instead of ρ_Σ and A/ρ_Σ, respectively. For every Congruence $\chi \in \mathtt{Con}(A)$ and a homomorphism $f : A \to B$, we define $f(\chi) := \{(f(a), f(a')) \mid (a,a') \in \chi\}$.

Note 1.1. A congruence χ_B on a subact B of an S-act A can be extended to a congruence on A. There is always the smallest extension χ_A of χ_B on A given by

$$(a,b) \in \chi_A \Longleftrightarrow \begin{cases} (a,b) \in \chi_B \\ a = b, \ \text{otherwise.} \end{cases}$$

Therefore we may consider each congruence $\chi_B \in \mathtt{Con}(B)$ as a congruence in $\mathtt{Con}(A)$ by identifying χ_B and χ_A. In particular, ∇_B can be considered as the generated Rees congruence by B, $\rho_B \in \mathtt{Con}(A)$.

In this paper whenever talking about a subclass $\mathbb{C}$ of S-acts, we assume that $\mathbb{C}$ is closed under taking isomorphic copies and $\mathbb{C}$ contains all trivial subacts. In the sequel we frequently use the notion of closedness of a subclass $\mathbb{C}$ of S-acts under a special property, such as homomorphic images, congruence extensions, general Rees extensions, subacts, products, coproducts, and inductive property defined in [22].

Although the radical notion for S-acts was introduced and investigated by R. Wiegandt [22], but, in order to employ this notion in **S-Act**, it seems necessary to define the radical in a more general manner. Here we reform the definition of Hoehnke radical given in [22] by the category theoretical view of radicals given in [7] and recall from [14] the following definition of Hoehnke radical in **S-Act**, which may also be called a normal Hoehnke radical.

Definition 1.2. A *normal Hoehnke radical* (or simply a *Hoehnke radical*) is an assignment $r : A \mapsto r(A)$, assigning to each S-act A a congruence $r(A) \in \mathtt{Con}(A)$ in such a way that

(i) r is functorial, that is, every homomorphism $f : A \to B$ induces the natural homomorphism $\tilde{f} : A/r(A) \to B/r(B)$ or equivantly f induces the homomorphism $f(r) : r(A) \to r(B)$ assigning each pair $(a, a') \in r(A)$ to $(f(a), f(a')) \in r(B)$. Note that $r(A)$ and $r(B)$ are, respectively, subacts of $A \times A$ and $B \times B$, since $r(A) \in \mathtt{Con}(A), r(B) \in \mathtt{Con}(B)$, and

(ii) $r(A/r(A)) = \Delta_{A/r(A)}$.

With every Hoehnke radical r, one can associate two classes of S-acts, namely *radical class* $\mathbb{R}_r$ and *semisimple class* $\mathbb{S}_r$, as follows:

$$\mathbb{R}_r = \{A \mid r(A) = \nabla_A\},$$
$$\mathbb{S}_r = \{A \mid r(A) = \Delta_A\}.$$

The class $\mathbb{R}_r$ is called the *radical class of* r (or r*-radical class*) and its members are called r-radical S-acts, and the class $\mathbb{S}_r$ is called the *semisimple class of* r (or r*-semisimple class*) and its members are called r-semisimple S-acts.

Definition 1.3. A Hoehnke radical r of S-acts is called a *Kurosh-Amitsur* radical if

(i) r(A) is a general Rees congruence, for all S-acts A, and

(ii) for every subact-$r(A)$-class B, $r(B) = \nabla_B$.

We recall, from [22], that a subclass $\mathbb{S}$ of S-acts is a semisimple class of a Kurosh-Amitsur radical r if and only if

1. $\mathbb{S}$ is closed under taking subacts,

2. $\mathbb{S}$ is closed under taking products,

3. $\mathbb{S}$ is closed under taking congruence extensions.

Also, a subclass $\mathbb{R}$ of S-acts is a Kurosh-Amitsur radical class of a radical r if and only if

1. $\mathbb{R}$ is homomorphically closed,
2. $\mathbb{R}$ has the inductive property,
3. $\mathbb{R}$ is closed under general Rees extensions.

Let us now define some radicals endowed with a certain property which is used in the sequel.

Definition 1.4. A radical r is said to be

(i) *hereditary* if for every S-act A and every subact B of A, $r(B) = r(A) \wedge \nabla_B$,

(ii) *pre-hereditary* if for every S-act A and $Y \leq X \in \Sigma_{r(A)}$, $Y \in \mathbb{R}_r$,

(iii) *weakly-hereditary* if, for every S-act A with a zero element θ and $X \in \Sigma_{r(A)}$ with $\theta \in X$, $X \in \mathbb{R}_r$,

(iv) *zero-hereditary* if, for every S-act A with a zero element θ and $Y \leq X \in \Sigma_{r(A)}$ with $\theta \in Y$, $Y \in \mathbb{R}_r$,

(v) *pre-Kurosh* if, for every S-act A and $X \in \Sigma_{r(A)}$, $X \in \mathbb{R}_r$.

Also, we recall from [9] that a *closure operator* C of the category $\mathcal{C}$ with respect to the class $\mathcal{M}$ of subobjects is a family $C = (C_X)_{X\in\mathcal{C}}$ of maps

$$\begin{array}{rccc} C_X : & \mathcal{M}/X & \rightarrow & \mathcal{M}/X \\ & (m : M \rightarrow X) & \rightsquigarrow & (C_X(m) : C_X(M) \rightarrow X), \end{array}$$

for every $m \in \mathcal{M}$, such that for every $X \in \mathcal{C}$

(Extension) $m \leq C_X(m)$, for all $m \in \mathcal{M}/X$,

(Monotonicity) if $m \leq m'$ in $\mathcal{M}/X$, then $C_X(m) \leq C_X(m')$,

(Continuity) $f(C_X(m)) \leq C_Y(f(m))$, for all $f : X \rightarrow Y$ in $\mathcal{C}$ and $m \in \mathcal{M}/X$.

For every $\mathcal{M}$-subobject $m : M \to X$ we have the following commutative diagram.

$$\begin{tikzcd} M \arrow[rr, "j_m"] \arrow[dr, "m"'] & & C_X(M) \arrow[dl, "C_X(m)"] \\ & X & \end{tikzcd}$$

A closure operator C is called *weakly hereditary* if $C_X(m) = C_{C_X(M)}(m)$, for every $\mathcal{M}$-subobject $m : M \to X$. Also, a closure operator C is called *idempotent* if $C_X(m) = C_X(C_X(m))$ for every $\mathcal{M}$-subobject $m : M \to X$.

An $\mathcal{M}$-subobject $m : A \to X$ of an object X is said to be C*-closed* if $C_X(m) = m$ and it is said to be C*-dense* if $C_X(m)$ is an isomorphism. An $\mathcal{M}$-morphism $m : B \to A$ is said C*-dense monomorphism* if $m(B)$ is C-dense in A.

The readers may consult [1, 6, 17] for general facts about category theory and universal algebra used in this paper. Here we also follow the notations and terminologies used there.

2 Injectivity relative to dense monomorphisms of a closure operator

Consider a subclass $\mathcal{M}$ of monomorphisms in a category $\mathcal{C}$ such that every object A in $\mathcal{C}$ has the $\mathcal{M}$-injective hull $(m_A; E_{\mathcal{M}}(A))$. Also let $C = (C_X)_{X \in \mathcal{C}}$ be a weakly hereditary and idempotent closure operator with respect to the class of monomorphisms and $\mathcal{D}$ be the class of C-dense monomorphisms. In this section, a monomorphism in $\mathcal{M} \cap \mathcal{D}$ is called $\mathcal{MD}$*-monomorphism* and it is shown that injectivity with respect to the $\mathcal{MD}$-monomorphisms, the so called $\mathcal{MD}$-injectivity, well-behaves. To do so, we begin with the following definition.

Definition 2.1. An object Q in the category $\mathcal{C}$ is said to be $\mathcal{MD}$*-injective* if for every $\mathcal{MD}$-monomorphism $m : B \to A$ and any morphism $f : B \to Q$ there exists a morphism $\bar{f} : A \to Q$ extending f, that is $\bar{f}$ completes the following commutative triangle.

$$\begin{tikzcd} B \arrow[r, "m"] \arrow[d, "f"'] & A \arrow[dl, dashed, "\bar{f}"] \\ Q & \end{tikzcd}$$

Lemma 2.2. *Let A be an object in the category $\mathcal{C}$ and $m : X \to Y$ be an $\mathcal{MD}$-monomorphism. Then, for any morphism $f : X \to A$, there exists a morphism $\bar{f} : Y \to C_{E_{\mathcal{M}}(A)}(A)$ which commutes the following square, in which $E_{\mathcal{M}}(A)$ is the $\mathcal{M}$-injective hull.*

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "f"'] & Y \arrow[d, "\bar{f}"] \\ A \arrow[r, "j_A"'] & C_{E_{\mathcal{M}}(A)}(A) \end{tikzcd}$$

Proof. Given a morphism $f : X \to A$ and an $\mathcal{MD}$-monomorphism $m : X \to Y$, the following commutative square follows from the $\mathcal{M}$-injectivity of $E_{\mathcal{M}}(A)$, in which $f' : Y \to E_{\mathcal{M}}(A)$ extends $m_A \circ f$ through m.

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "f"'] & Y \arrow[d, "f'"] \\ A \arrow[r, "m_A"'] & E_{\mathcal{M}}(A) \end{tikzcd}$$

But, by Diagonalization Lemma (see 2.4 in [9]) and the C-density of m, there is a uniquely determined morphism $\bar{f} : Y \to C_{E_{\mathcal{M}}(A)}(A)$ rendering the following diagram and we are done.

$$\begin{tikzcd} X \arrow[rr, "m"] \arrow[dd, "f"'] & & Y \arrow[dl, "\bar{f}"'] \arrow[dd, "f'"] \\ & C_{E_{\mathcal{M}}(A)}(A) \arrow[dr, "C_{E_{\mathcal{M}}(A)}(m_A)"] & \\ A \arrow[ur, "j_A"] \arrow[rr, "m_A"'] & & E_{\mathcal{M}}(A) \end{tikzcd}$$

$\square$

Recall the following types of essential extensions usually used in the literature.

Definition 2.3. Given a subclass $\mathcal{M}$ of monomorphisms, an $\mathcal{M}$-monomorphism $m : M \to X$ will be called:

(i) $\mathcal{M}_{e_1}$*-essential* if $f \circ m \in \mathcal{M}$ implies $f \in \mathcal{M}$, for every morphism f.

(ii) $\mathcal{M}_{e_2}$*-essential* if $f \circ m$ being monomorphism implies that f is a monomorphism.

(iii) $\mathcal{M}_{e_3}$*-essential* if $f \circ m \in \mathcal{M}$ implies that f is a monomorphism, for every morphism f.

Notation 2.4. We denote the class of $\mathcal{M}_{e_i}$-essential monomorphisms by $\mathcal{M}^*_{e_i}$, for $i \in \{1, 2, 3\}$.

Definition 2.5. An $(\mathcal{MD})_{e_i}$*-injective hull* of an object A, $i = 1, 2, 3$, is a pair $(m_A, E_{(\mathcal{MD})_{e_i}}(A))$ consisting of an $\mathcal{MD}$-injective object $E_{(\mathcal{MD})_{e_i}}(A)$ and an $(\mathcal{MD})_{e_i}$-essential monomorphism $m_A : A \to E_{(\mathcal{MD})_{e_i}}(A)$, for $i = 1, 2, 3$.

Proposition 2.6. *Let A be an object in the category $\mathcal{C}$ and $i \in \{1, 2, 3\}$. Then $(j_{m_A}, C_{E_{\mathcal{M}}(A)}(A))$ is the $(\mathcal{MD})_{e_i}$-injective hull of A, if $\mathcal{M}^*_{e_i}$ is left cancelable for monomorphisms, that is $nm \in \mathcal{M}^*_{e_i}$ implies $m \in \mathcal{M}^*_{e_i}$, when n and m are monomorphisms.*

Proof. Since C is an idempotent closure operator, $C_{E_{\mathcal{M}}(A)}(A)$ is an $\mathcal{MD}$-injective object, by Lemma 2.2. So to prove, we show that $j_{m_A} : A \to C_{E_{\mathcal{M}}(A)}(A)$ is an $(\mathcal{MD})_{e_i}$-essential monomorphism where m_A is an $\mathcal{M}_{e_i}$-essential monomorphism.

$$\begin{tikzcd}
A \arrow[rr, "j_{m_A}"] \arrow[dr, "m_A"'] & & C_{E_{\mathcal{M}}(A)}(A) \arrow[dl, "C_{E_{\mathcal{M}}(A)}(m_A)"] \\
& E_{\mathcal{M}}(A) &
\end{tikzcd}$$

Indeed, by the weakly heredity of the closure operator C, j_{m_A} is a C-dense monomorphism and if $j_{m_A} \circ f$ is a C-dense monomorphism, then f is C-dense monomorphism, for every morphism f, see Section 2.3 of [9]. Also, since $\mathcal{M}^*_{ei}$ is left cancelable for monomorphisms, the $\mathcal{M}_{e_i}$-essentially of m_A implies that j_{m_A} is an $\mathcal{M}_{e_i}$-essential monomorphism. So, $j_{m_A} : A \to C_{E_{\mathcal{M}}(A)}(A)$ is an $(\mathcal{MD})_{e_i}$-essential monomorphism. $\square$

Corollary 2.7. *If the class $\mathcal{M}^*_{e_i}$ is left cancellable for monomorphisms, then $(j_{m_A}, C_{E_{\mathcal{M}}(A)}(A))$ is the minimal $\mathcal{MD}$-injective extension of A and the maximal $(\mathcal{MD})_{e_i}$-essential extension of A, for every object $A \in \mathcal{C}$ and $i \in \{1,2,3\}$.*

Proof. The result easily follows from the following commutative triangles in which (q,Q) is an $\mathcal{MD}$-injective extension of A and (m,M) is an $(\mathcal{MD})_{e_i}$-essential extension of A.

$$\begin{tikzcd} A \arrow[r, "m"] \arrow[d, "j_{m_A}"'] & M \arrow[dl, "\overline{j_{m_A}}"] \\ C_{E_{\mathcal{M}}(A)}(A) & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "j_{m_A}"] \arrow[d, "q"'] & C_{E_{\mathcal{M}}(A)}(A) \arrow[dl, "\bar{q}"] \\ Q & \end{tikzcd}$$

□

With Lemma 2.7 of [11] in mind, we also have the following corollary.

Corollary 2.8. *Let the class $\mathcal{M}^*_{e_i}$ be left cancellable for monomorphisms. Then the following conditions are equivalent, for any object A and $i \in \{1,2,3\}$.*

(1) *A is $\mathcal{MD}$-injective.*
(2) *A is $\mathcal{MD}$-absolute retract.*
(3) *A has no proper $(\mathcal{MD})_{e_i}$-essential extension.*

Proof. (1) $\Leftrightarrow$ (2), immediately, follows from Lemma 2.7 of [11] and (1) $\Leftrightarrow$ (3) follows from Proposition 2.6 and Corollary 2.7. □

Based on the results and discussions presented in this section, we have the following conclusions.

Corollary 2.9. *Let $\mathcal{M}$ be a subclass of monomorphisms in a category $\mathcal{C}$ such that the category $\mathcal{C}$ has enough $\mathcal{M}$-injective hull and $\mathcal{M}^*_{e_i}$ is left cancellable for monomorphisms, where $i \in \{1,2,3\}$. Also, let C be an idempotent and weakly hereditary closure operator with respect to the class of monomorphisms. Then the notion of $\mathcal{MD}$-injectivity well-behaves.*

Moreover, if the category $\mathcal{C}$ has enough injective hull and $\mathcal{M}$ can be extended to the class of dense monomorphisms of an idempotent and weakly *hereditary closure operator with respect to the class of monomorphisms, then the notion of $\mathcal{M}$-injectivity well-behaves.*

Proof. The result, immediately, follows from Proposition 2.6 and corollaries 2.7 and 2.8. Moreover, If $\mathcal{M}$ can be extended to the class of dense monomorphisms of an idempotent and weakly hereditary closure operator C, then $\mathcal{M} = \mathcal{M} \cap \mathcal{D}$ in which $\mathcal{D}$ is the class of C-dense monomorphisms. That is, $\mathcal{M}$ coincides with the class $\mathcal{MD}$-monomorphisms. □

Example 2.10. In the categories **R-mod** and **S-Act**, the class of essential monomorphisms is left cancellable for monomorphisms. So, by the above corollary, we have the following examples.

(1) Given an idempotent radical r over the category **R-mod**, the closure operator c^r defined by $c^r_M(N) = \pi^{-1}(r(M/N)$, for every $N \leq M \in$ **R-mod** and the canonical epimorphism $\pi : M \to M/N$, is both weakly hereditary and idempotent, see Chapter 3 of [9]. So r-injectivity in the category **R-mod** well-behaves and $c^r_{(E(M))}(M)$ is the r-injective hull of an R-module M when $E(M)$ is the injective hull of M, see [8].

(2) Given a weakly hereditary Hoehnke radical r over the category **S-Act**, the closure operator c^r defined by $c^r_A(B) = \pi^{-1}(r([B]_{r(A/B)})$, for every $B \leq A \in$ **S-Act** and the canonical epimorphism $\pi : A \to A/B$, is both weakly hereditary and idempotent, see [15]. So, r-injectivity is well-behavior and $c^r_{(E(A))}(A)$ is the r-injective hull of an S-act A when $E(A)$ is the injective hull of A.

(3) Take C^p to be the closure operator defined by

$$C^p(A) = \{b \in B \mid \exists a \in A, \rho_b = \rho_a\}$$

in which $\rho_x : S \to A$ is defined by $\rho_x(s) = sx$, for each $s \in S$. In [10], it is shown that C^p is both idempotent and weakly hereditary. So, for an S-act A, $C^p_{(E(A))}(A)$ is the C^p-injective hull of A.

We end this section with giving a criterion for the left cancellability of $\mathcal{M}^*_{e_i}$ for monomorphisms, for $i \in \{1, 2, 3\}$.

Definition 2.11. Let $\mathcal{M}$ be a subclass of monomorphisms of a category $\mathcal{C}$ which has pushouts. We say that *pushouts transfer $\mathcal{M}$-monomorphisms* whenever, for the pushout diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ Z & \xrightarrow{k} & W \end{array}$$

with $f \in \mathcal{M}$, we have $k \in \mathcal{M}$.

Theorem 2.12. *Let a category $\mathcal{C}$ have pushouts and $\mathcal{M}$ be a subclass of monomorphisms. Then $\mathcal{M}^*_{e_i}$ is left cancellable for monomorphisms, for $i \in \{1, 2, 3\}$, if $\mathcal{M}$ is right cancellable for monomorphisms, closed under composition and pushouts transfer $\mathcal{M}$-monomorphisms.*

Proof. Let f and g be monomorphisms with $m = gf \in \mathcal{M}^*_{e_1}$. We show that f is $\mathcal{M}_{e_1}$-essential monomorphism. To do so, let h be a morphism such that hf is $\mathcal{M}$-monomorphism. Taking the pushout of g, h we get morphisms p, q with $pg = qh$.

$$\begin{array}{ccccc} A & \xrightarrow{f} & X & \xrightarrow{h} & Y \\ & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle q} \\ & & Z & \xrightarrow{p} & W \end{array}$$

But since $m = gf$ is an $\mathcal{M}$-monomorphism, the right cancellability of $\mathcal{M}$ implies g and consequently q are $\mathcal{M}$-monomorphisms. So qhf is an $\mathcal{M}$-monomorphism, by the closedness of $\mathcal{M}$ under composition. Thus $pgf = pm$ is an $\mathcal{M}$-monomorphism. Now, we can infer from the $\mathcal{M}_{e_1}$-essentially of m that p is an $\mathcal{M}$-monomorphism. Hence the closedness of $\mathcal{M}$ under composition implies that $pg = qh$ is an $\mathcal{M}$-monomorphism. So h is a monomorphism. Therefore, since $\mathcal{M}$ is right cancellable for monomorphisms, $h \in \mathcal{M}$ follows from $hf \in \mathcal{M}$. The left cancellability of $\mathcal{M}^*_{e_2}$ and $\mathcal{M}^*_{e_3}$ are proved completely analogues. □

3 (r, t)-injectivity

In this section, we introduce the notion of (r, t)-injectivity, with respect to two Hoehnke radicals r and t over the category **S-Act**, similar to what

Beachy defined for R-modules in [4]. We also give some characterization of the notion of (r,t)-injectivity.

Given a Hoehnke radical r, a subact B of an S-act A is said to be *r-dense* if $A/B \in \mathbb{R}_r$. Similarly, a congruence χ on A is said to be *r-dense* if $A/\chi \in \mathbb{R}_r$. A monomorphism $\iota : B \longrightarrow A$ is said *r-monomorphism* if $\iota(B)$ is r-dense in A.

Definition 3.1. Let r and t be two Hoehnke radicals over **S-Act**.

(i) A pair $(m : B \to A, e : B \to C)$ of homomorphisms is said to be *(r,t)-pair* if m is an r-monomorphism with the non-empty domain and $m(\ker(e)) \cup \Delta_A$ is a t-dense congruence on A.

(ii) An S-act Q is said to be *(r,t)-injective* if every (r,t)-pair $(m : B \to A, f : B \to Q)$ can be completed to the following commutative triangle.

$$\begin{tikzcd} B \arrow[r, "m"] \arrow[d, "f"'] & A \arrow[dl, dashed, "\bar{f}"] \\ Q & \end{tikzcd}$$

Theorem 3.2. *Let r and t be two Hoehnke radicals. Then a subact P of an (r,t)-injective S-act Q is (r,t)-injective if $Q/P \in \mathbb{S}_r$.*

Proof. Suppose $Q/P \in \mathbb{S}_r$ and consider the commutative diagram

$$\begin{tikzcd} B \arrow[r, "m"] \arrow[d, "f"'] & A \arrow[d, dashed, "\bar{f}"] \\ P \arrow[r, hook] & Q \end{tikzcd}$$

in which (m,f) is an (r,t)-pair. Then the homomorphism $f' : A/m(B) \to Q/P$ defined by $a/m(B) \mapsto \bar{f}(a)/P$ arises. Now since $A/m(B) \in \mathbb{R}_r$ and $\mathbb{R}_r$ is closed under homomorphic images, $f'(A/m(B)) \in \mathbb{R}_r$. Also, we have $Q/P \in \mathbb{S}_r$. So the closedness of $\mathbb{S}_r$ under subacts implies that $f'(A/m(B)) \in \mathbb{S}_r$ and therefore $f'(A/m(B)) \in \mathbb{S}_r \cap \mathbb{R}_r$. But $\mathbb{S}_r \cap \mathbb{R}_r$ consists of the trivial S-acts, so f' is a zero homomorphism. The commutativity of the above square ensures that $\bar{f}(A) \subseteq P$. That is, $\bar{f}$ is an extension of f from A to P, and we are done. $\square$

It is worth noting that injectivity of Q relative to the (r,t)-pairs of the form $(A_0 \hookrightarrow A, f : A_0 \to Q)$, implies injectivity of Q relative to all (r,t)-pairs. Indeed, for every (r,t)-pair $(m : B \to A, f : B \to Q)$, considering the following commutative diagram, one can define $f' : m(B) \to Q$, by $f'(m(b)) = f(b)$, for every $b \in B$, and extend f' to $\bar{f}$ through the inclusion map. Hence f is extended to $\bar{f}$ through m.

$$\begin{tikzcd}
B \arrow[r, "m"] \arrow[dr, "f"'] & m(B) \arrow[r, hook] \arrow[d, "f'"] & A \arrow[dl, dashed, "\bar{f}"] \\
& Q &
\end{tikzcd}$$

In the following theorem, we use the above fact and give an analogous Skornjakov criterion for (r,t)-injective S-acts when r is zero-hereditary.

Theorem 3.3 (Skornjakov)**.** *Let r be a zero-hereditary Hoehnke radical and t be a hereditary Hoehnke radical over* **S-Act***. Also, let Q be an S-act with a zero element θ. Then Q is (r,t)-injective if and only if, for every (r,t)-pair $(A_0 \hookrightarrow A, f : A_0 \to Q)$ in which A is a cyclic S-act, there exists $\bar{f} : A \to Q$ in such a way that $\bar{f}|_{A_0} = f$.*

Proof. For the nontrivial way, we follow the standard proof of Skornjakov theorem and show that, for every (r,t)-pair $(B \hookrightarrow A, f : B \to Q)$, f can be extended to A. So we take the poset

$$T = \{h : C \to Q \mid B \leq C \leq A, \text{ and } h|_B = f\}$$

together with the partial order

$$h_1 \leq h_2 \Leftrightarrow \mathtt{Dom}(h_1) \leq \mathtt{Dom}(h_2) \text{ and } h_2|_{\mathtt{Dom}(h_1)} = h_1.$$

We should note that, for every $h : C \to Q$ in T, since $B \leq C \leq A$ and B is r-dense in A, C is so. Also $\ker(h) \cup \Delta_A$ is a t-dense congruence on A, for every $h \in T$, because $A/(\ker(h) \cup \Delta_A)$ is the homomorphic image of $A/(\ker(f) \cup \Delta_A)$ and $\mathbb{R}_t$ is homomorphically closed. Also one can easily see that every ascending chain $\{h_i : C_i \to Q\}_{i \in I}$ of $(T, \leq)$ has the upper bound $h : \bigcup_{i \in I} C_i \to Q$ in which $h(x) = h_i(x)$; where $x \in \mathtt{Dom}(h_i)$. Hence T has a maximal element such as $h : A_1 \to Q$, by Zorn's Lemma. Now we show that

$A = A_1$. To do so, suppose on the contrary that $A_1 \lneq A$. Then there exists $a \in A \setminus A_1$ for which we define $D = A_1 \cap Sa$. If $D = \emptyset$, then

$$\begin{array}{rcl} \bar{f}: \ A & \longrightarrow & Q \\ a & \mapsto & \begin{cases} h(a) & a \in A_1 \\ \theta & a \in A \setminus A_1 \end{cases} \end{array}$$

is an extension of f which commutes the following diagram which is a contradiction:

$$\begin{tikzcd} B \arrow[r, hook] \arrow[d, "f"'] & A \arrow[dl, dashed, "\bar{f}"] \\ Q & \end{tikzcd}$$

If $D \neq \emptyset$, then D is an r-dense subact of Sa because the kernel of a homomorphism $k : Sa \to A/A_1$ defined by $k(sa) = sa/A_1$ is ρ_D. So, by Homomorphism Theorem for S-acts, Sa/D is isomorphic to a subact H of A/A_1. Now, since r is a zero-hereditary Hoehnke radical and H is a subact with a zero element of the r-radical S-act A/A_1, we have $r(H) = \nabla_H$. That is $r(Sa/D) = \nabla_{Sa/D}$. Also, by taking $g = h|_D : D \to Q$, $\ker(g) \cup \Delta_{S_a}$ is a t-dense congruence on Sa. Indeed, $\ker(g) = \ker(h) \cap \nabla_D = \ker(h) \cap \nabla_{Sa}$ and $Sa/(\ker(g) \cup \Delta_{Sa})$ is isomorphic to a subact H' of $A/(\ker(h) \cup \Delta_A)$. Now, since $\ker(h) \cap \Delta_A$ is t-dense on A, $A/(\ker(h) \cup \Delta_A)$ is t-radical and hence the heredity of t implies that $Sa/(\ker(g) \cup \Delta_{Sa})$ is t-radical which proves that $\ker(g) \cup \Delta_{Sa}$ is t-dense on Sa. Therefore, our assumption implies the existence of an extension $\bar{g} : Sa \to Q$ of g. Thus this means that

$$\begin{array}{rcl} \bar{h}: \ A_1 \cup Sa & \longrightarrow & Q \\ x & \mapsto & \begin{cases} h(x) & x \in A_1 \\ s\bar{g}(a) & x = sa \in Sa \end{cases} \end{array}$$

is an extension of h and it contradicts the maximality of h. So $A_1 = A$ and we are done. $\square$

Definition 3.4. Let r be a Hoehnke radical and $A \in$ **S-Act**. Then an S-act Q is said to be A-*injective* if each homomorphism $f : B \to Q$, in which B is a non-empty subact of A, can be extended to A. Also, an S-act Q is said to be (r, A)-*injective* if each homomorphism $f : B \to Q$, in which B is a non-empty r-dense subact of A, can be extended to A.

Lemma 3.5. *Let r be a Hoehnke radical and A be an r-radical S-act. Then an S-act Q is (r,A)-injective if and only if Q is an A-injective S-act*

Proof. The result follows from the fact that every subact of an r-radical S-act A is r-dense in A. □

Theorem 3.6. *Let r be a Hoehnke radical and $A \in$ **S-Act**. Then the following conditions are equivalent:*

(1) *An S-act Q is (r,A)-injective.*

(2) *If $f^{-1}(Q) = \{a \in A \mid f(a) \in Q\}$ is r-dense in A, for a homomorphism $f : A \to E_r(Q)$, then there exists a homomorphism $\bar{f} : A \to Q$ such that*

$$(f^{-1}(Q), \hookrightarrow) = Eq(f : A \to E_r(Q), \bar{f} : A \to Q \hookrightarrow E_r(Q)).$$

Proof. (1) $\Rightarrow$ (2) Let $f : A \to E_r(Q)$ be a homomorphism such that $f^{-1}(Q)$ is an r-dense subact of A. Then $f|_{f^{-1}(Q)}$ can be extended to $\bar{f} : A \to Q$. So Proposition 2.2.10 of [17] implies that $i : f^{-1}(Q) \to A$ is an equalizer of f and $\bar{f}$.

(2) $\Rightarrow$ (1) Let B be an r-dense subact of A and $f : B \to Q$ be a homomorphism. Then there exists an extension $f' : A \to E_r(Q)$ of f. So, since B is an r-dense subact of A and $B \leq f'^{-1}(Q)$, $f'^{-1}(Q)$ is an r-dense subact of A. Hence, by hypothesis, there exists $\bar{f} : A \to Q$ such that the inclusion map $f'^{-1}(Q) \hookrightarrow A$ is an equalizer of f' and $\bar{f}$. Therefore $\bar{f}|_B = f$ and we are done. □

Theorem 3.7. *Given a hereditary Hoehnke radical r, if $f : A \to E_r(Q)$ is a homomorphism with $\ker(f) \leq \rho_{f^{-1}(Q)}$, then $f^{-1}(Q)$ is an r-dense subact of A.*

Proof. Let $f : A \to E_r(Q)$ be a homomorphism with $\ker(f) \leq \rho_{f^{-1}(Q)}$. Then the map $m : A/f^{-1}(Q) \to E_r(Q)/Q$ defined by $m(a/f^{-1}(Q)) = f(a)/Q$ is a monomorphism. So, the heredity of r implies that $A/f^{-1}(Q) \in \mathbb{R}_r$ and we are done. □

Theorem 3.8. *Let r and t be two Hoehnke radicals over* **S-Act**. *An S-act Q is (r,t)-injective if and only if Q is (r,A)-injective, for every $A \in \mathbb{R}_t$.*

Proof. The necessity, immediately follows from the fact that every congruence on a t-radical S-act is t-dense. To prove the sufficiency, suppose Q is

(r, A)-injective S-act for every $A \in \mathbb{R}_t$. Also, assume $(B_0 \hookrightarrow B, g : B_0 \to Q)$ is an (r,t)-pair. Then the t-density of $\ker(g) \cup \Delta_B$ on B implies that $B/(\ker(g) \cup \Delta_B) \in \mathbb{R}_t$ and so the homomorphism $g' : B_0/\ker(g) \longrightarrow Q$ defined by $g'(b/\ker(g)) = g(b)$, for every $b \in B_0$, can be extended to a homomorphism $\bar{g'} : B/(\ker(g) \cup \Delta_B) \longrightarrow Q$, by hypothesis. Now, One can easily cheek the map $\bar{g} : B \to Q$ defined by $\bar{g}(b) = \bar{g'}(b/(\ker(g) \cup \Delta_B))$ is an extension of g. Therefore Q is (r,t)-injective. □

4 (r, t)-injectivity and t-subacts

In this section, we introduce a class of subacts which play a considerable role in (r,t)-injectivity.

Definition 4.1. Given an S-act A, the union of all r-radical subacts of A is said to be *r-subact of A* and is denoted by R^r_A.

Theorem 4.2. *Let r and t be two Hoehnke radicals and Q be an (r,t)-injective S-act. Then, for every (r,t)-pair $(m : B \to A, f : B \to Q)$, the image of every extension of f through m is a subact of R^t_Q. Moreover, R^t_Q is an (r,t)-injective S-act.*

Proof. Let $\bar{f} : A \to Q$ be an extension of f through m, for a given (r,t)-pair $(m : B \to A, f : A \to Q)$, ($(r,t)$-injectivity of Q ensures the existence of $\bar{f}$).

$$\begin{tikzcd} B \arrow[r, "m"] \arrow[d, "f"'] & A \arrow[dl, "\bar{f}"] \\ Q & \end{tikzcd}$$

Since $m(\ker(f)) \subseteq \ker(\bar{f})$, $A/\ker(\bar{f})$ is the homomorphic image of $A/(m(\ker(f) \cup \Delta_A) \in \mathbb{R}_t$. Now the closedness of $\mathbb{R}_t$ under homomorphic images implies $A/\ker(\bar{f}) \in \mathbb{R}_t$. Thus $\bar{f}(A) \cong A/\ker(\bar{f}) \leq R^t_Q$.

Moreover, since every (r,t)-pair $(m : B \to A, f : A \to R^t_Q)$ can be considered as an (r,t)-pair $(m : B \to A, f : A \to Q)$, R^t_Q is (r,t)-injective and we are done. □

Theorem 4.3. *Given two Hoehnke radicals r and t over* **S-Act***, a t-semisimple S-act A is (r,t)-injective if $E_r(A)$ is a t-semisimple S-act.*

Proof. To prove, we show that, for every (r,t)-pair $(m : X \to Y, f : X \to A)$, f is a zero homomorphism. Indeed, the r-injectivity of $E_r(A)$ implies that the following commutative diagram is completed by a homomorphism $f' : Y \to E_r(A)$:

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "f"'] & Y \arrow[d, "f'"] \\ A \arrow[r, hook] & E_r(A) \end{tikzcd}$$

Also, by Theorem 4.2, the homomorphic image of Y under f' is a subact of $R^t_{E_r(A)}$ and, by hypothesis, $R^t_{E_r(A)}$ is a trivial S-act. So f' is a zero homomorphism. Hence f is a zero homomorphism and we are done. $\square$

We recall, from Theorem 7.4 of [15], that a Kurosh-Amitsur radical r is hereditary if and only if $\mathbb{R}_r$ is closed under r-injective hulls. So, Theorem 4.3 implies the following corollary.

Corollary 4.4. *Given two Hoehnke radicals r and t, if t is a hereditary Kurosh-Amitsur radical, then every t-semisimple S-act is (r,t)-injective.*

Theorem 4.5. *Let r be a hereditary Hoehnke radical and t be a pre-Kurosh radical. Then the following conditions are equivalent, for an S-act Q:*

(1) *The S-act Q is (r,t)-injective.*

(2) *For every $A \in \Sigma_{t(E_r(Q))}$, we have $A \leq Q$.*

(3) $\Sigma_{t(Q)} = \Sigma_{t(E_r(Q))}$.

(4) $R^t_{E_r(Q)} = R^t_Q$.

(5) *Each homomorphism $f : B \to Q$ can be extended to a homomorphism $\bar{f} : A \to Q$, for every S-act A with $A = B \cup (A \cap R^t_{E_r(A)})$.*

Proof. (1) $\Rightarrow$ (2) First we note that, since Q is large in $E_r(Q)$, by Lemma 1.15 of [17], $\rho_Q \cap \rho_A \neq \Delta_{E_r(Q)}$ and hence $A \cap Q$ is nonempty, for every $A \in \Sigma_{t(E_r(Q))}$. Also, since every congruence χ on A can be extended to a congruence $\bar{\chi}$ on $E_r(Q)$, see Note 1.1, the largeness of Q in $E_r(Q)$ implies $\rho_Q \cap \bar{\chi} \neq \Delta_{E_r(Q)}$. Hence there exist $a \neq b \in A$ such that $(a,b) \in \rho_Q \cap \bar{\chi}$. Therefore, by definition of $\bar{\chi}$, $(a,b) \in \chi \cap \rho_{A \cap Q}$. That is, by Lemma 1.15 of [17], $Q \cap A$ is large in A. Now the heredity of r and the r-density of Q in $E_r(Q)$ implies that $A/(Q \cap A) \in \mathbb{R}_r$. So, the pair of inclusion maps $(\iota_1 : Q \cap A \to A, \iota_2 : Q \cap A \to Q)$ is an (r,t)-pair. Hence the existence of a homomorphism $i : A \to Q$ with $i \circ \iota_1 = \iota_2$ follows from the (r,t)-injectivity of Q. But, since $Q \cap A$ is large in A and ι_2 is a monomorphism, i is a monomorphism. So there exists a subact A' of Q such that $A' \cong A$ and $Q \cap A \leq A'$. Hence the closedness of $\mathbb{R}_t$ under homomorphic images implies that there is $B \in \Sigma_{t(E_r(Q))}$ with $A' \leq B$. But $\Sigma_{t(E_r(Q))}$ is a set of disjoint subacts of $E_r(Q)$. Therefore $A' = B = A$. That is $A \leq Q$.

(2) $\Rightarrow$ (3) First we show that $\Sigma_{t(E_r(Q))} \subseteq \Sigma_{t(Q)}$. To do so, let $A \in \Sigma_{t(E_r(Q))}$. Then $A \leq Q$, by hypothesis. Also since t is a pre-Kurosh radical, we have $A \in \mathbb{R}_t$. So, there exists $A' \in \Sigma_{t(Q)}$ such that $A \leq A'$. Thus A' is a t-radical S-act. Hence $A' \leq Q \leq E_r(Q)$ implies that there exists $A'' \in \Sigma_{t(Q)}$ such that $A' \leq A''$. But $\Sigma_{t(E_r(Q))}$ is a set of disjoint subacts of $E_r(Q)$, so we have $A' = A'' = A$. By an analogous argument, one can prove $\Sigma_{t(Q)} \subseteq \Sigma_{t(E_r(Q))}$.

(3) $\Rightarrow$ (4) Since t is a pre-Kurosh radical, $\Sigma_{t(Q)} = \Sigma_{t(E_r(Q))}$ immediately implies that $R^t_{E_r(Q)} = R^t_Q$.

(4) $\Rightarrow$ (5) Let $A = B \cup (A \cap R^t_{E_r(A)})$ and $f : B \to Q$ be a homomorphism. Then there exists a homomorphism $f' : E_r(A) \to E_r(Q)$ which commutes the following square:

$$
\begin{tikzcd}
A \arrow[r, hook] \arrow[d, "f"'] & E_r(A) \arrow[d, "f'"] \\
Q \arrow[r, hook] & E_r(Q)
\end{tikzcd}
$$

So, we have

$$
\begin{aligned}
f'(A) &= f'(B \cup (A \cap R^t_{E_r(A)})) \\
&\leq f'(B) \cup f'(R^t_{E_r(A)}) \\
&\leq f(B) \cup R^t_{f'(E_r(A))} \\
&\leq f(B) \cup R^t_{E_r(Q)} \\
&= f(B) \cup R^t_Q \qquad\qquad (*) \\
&\leq Q
\end{aligned}
$$

in which the equation $(*)$ follows from the hypothesis. Hence $f'|_A : A \to Q$ is an extension of f and we are done.

(5) $\Rightarrow$ (1) Let $f : B \to Q$ be a homomorphism and $m : B \to A$ be an r-monomorphism with $A \in \mathbb{R}_t$. Then $A = m(B) \cup (A \cap R^t_{E_r(A)})$ follows from

$m(B) \leq A \leq R^t_{E_r(A)}$. Now using the hypothesis, one can get an extension $\bar{f} : A \to Q$ of f with $\bar{f}m = f$ and we are done. $\square$

The converse of Theorem 4.3, for hereditary Hoehnke radical r and Kurosh-Amitsur radical t, follows from of the above theorem.

Theorem 4.6. *Let r be a hereditary Hoehnke radical and t be a Kurosh-Amitsur radical. Then a t-semisimple S-act A is (r,t)-injective if and only if $E_r(A)$ is a t-semisimple S-act.*

Proof. One way is Theorem 4.3. For the converse, let A be (r,t)-injective. Then, by Theorem 4.5, $\Sigma_{t(A)} = \Sigma_{t(E_r(A))}$. But A is t-semisimple and so it does not contain any nontrivial t-radical subact. Hence $\Sigma_{t(E_r(A))} = \Sigma_{t(A)} = \emptyset$, which means, $E_r(A)$ is a t-semisimple S-act. $\square$

Theorem 4.7. *Let r be a Hoehnke radical and t be a Kurosh-Amitsur radical over **S-Act**. Then the following conditions are equivalent.*

(1) *The Hoehnke radical t is hereditary.*

(2) *The r-injective hull $E_r(A)$ is t-semisimple if and only if A is a t-semisimple S-act.*

(3) *Every $A \in \mathbb{S}_t$ is (r,t)-injective.*

(4) *The Hoehnke radical t is pre-hereditary.*

Proof. (1) $\Rightarrow$ (2), In Theorem 7.4 from [15] we have shown that a Kurosh-Amitsur radical t is hereditary if and only if $\mathbb{S}_t$ is closed under injective hulls. Here, since $\mathbb{S}_t$ is closed under subacts and, for every S-act A, $E_r(A) \leq E(A)$, $\mathbb{S}_t$ is closed under r-injective hulls.

(2) $\Rightarrow$ (3) follows from Theorem 4.3.

(3) $\Rightarrow$ (4) Given an S-act B and $Y \leq X \in \Sigma_{t(B)}$, we prove that Y is a t-radical S-act. Indeed, $Y/t(Y)$ is an (r,t)-injective S-act, by hypothesis. So there exists a homomorphism $\bar{\pi} : X \to Y/t(Y)$ which commutes the following triangle, in which $\pi : Y \to Y/t(Y)$ is the canonical homomorphism.

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d, "\pi"'] & X \arrow[dl, dashed, "\bar{\pi}"] \\ Y/t(Y) & \end{tikzcd}$$

But, since t is a Kurosh-Amitsur radical, $X \in \mathbb{R}_t$. So $Y/t(Y) \in \mathbb{S}_t$ implies that $\bar{\pi}$ is a zero homomorphism and hence π is a zero homomorphism. That is, $Y/t(Y)$ is a trivial S-act, or equivalently $Y \in \mathbb{R}_t$ and we are done.

(4) $\Rightarrow$ (1) It is straightforward to check, see Figure 1 of [15]. $\square$

5 Well-behavedness of (r,t)-injectivity

We begin this section with the following criterion for (r,t)-injectivity. Then, we investigate product and coproduct of (r,t)-injective S-acts. Finally, we prove that for a hereditary Hoehnke radical r and a pre-Kurosh radical t, (r,t)-injectivity is well-behavior.

Theorem 5.1. *Let r be a zero-hereditary Hoehnke radical and t be a hereditary Hoehnke radical over* **S-Act**. *Also, let Q be an S-act with a zero element θ. Then Q is (r,t)-injective if and only if, every inclusion map $i : Q_0 \longrightarrow Q$ can be extended through every r-essential cyclic t-radical extension. More explicitly, Q is (r,t)-injective if and only if every diagram*

$$\begin{tikzcd} Q_0 \arrow[r, "m"] \arrow[d, hook, "i"'] & A \\ Q & \end{tikzcd}$$

for which i is an inclusion map and m is an r-essential monomorphism and A is a cyclic t-radical S-act can be completed by a homomorphism $g : A \to Q$.

Proof. One way is clear. For the converse, using Theorem 3.3, let X be an r-dense subact of a cyclic S-act Y and $f : X \to Q$ be a homomorphism such that $Y/(\ker(f) \cup \Delta_Y) \in \mathbb{R}_t$. Then, by Theorem 3.9 of [15], there exists $\kappa \in \texttt{Con}(Y)$ such that $X/\ker(f) \cong X/\kappa_{|X}$ is r-large in Y/κ. So, since $\ker(f) \leq \kappa$, the closedness of $\mathbb{R}_t$ under homomorphic images implies that $Y/\kappa \in \mathbb{R}_t$. Thus we take the subact Q_0 of Q to be $Q_0 \cong X/\ker(f)$ and a cyclic t-radical extension A of Q_0 to be $A \cong Y/\kappa$. Now, by hypothesis, there is a homomorphism $\bar{f}$ which commutes the diagram

$$
\begin{tikzcd}
X \arrow[rr, hook] \arrow[d, "\pi_X"'] \arrow[dddr, "f"] & & Y \arrow[d, "\pi_Y"] \\
X/\ker(f) \arrow[rr, hook] \arrow[d, leftrightarrow, "\psi"'] & & Y/\kappa \arrow[d, leftrightarrow, "\varphi"] \\
Q_0 \arrow[rr, hook] \arrow[dr, hook, "i"'] & & A \arrow[dl, "\bar{f}"] \\
& Q &
\end{tikzcd}
$$

So, $\bar{f} \circ \varphi \circ \pi_Y$ is an extension of f. $\square$

Theorem 5.2. *Given an (r,t)-injective S-act Q, every $T \in \Sigma_{t(Q)}$ is (r,t)-injective.*

Proof. By Theorem 3.8, let B be an r-dense subact of $A \in \mathbb{R}_t$ and $f : B \to T$ be a homomorphism. Then, for the inclusion map $\iota : T \to Q$, $\iota \circ f$ can be extended through m to a homomorphism $\bar{f} : A \to Q$. Hence the closedness of $\mathbb{R}_t$ under homomorphic images implies that the homomorphic image of A under $\bar{f}$ is a t-radical subact of Q containing T. So $\bar{f}(A)$ is contained in a member X of $\Sigma_{t(Q)}$. But, since $\Sigma_{t(Q)}$ is a set of disjoint subacts of Q and $X, T \in \Sigma_{t(Q)}$ have nonempty intersection, $f(B) \leq X \cap T \neq \emptyset$, we have $T = X$. Hence the homomorphic image of A under $\bar{f}$ is a subact of T. Therefore T is (r,t)-injective. $\square$

Theorem 5.3. *Given two Hoehnke radicals r and t,*

(1) *If $\mathbb{R}_r \cap \mathbb{R}_t$ consists of trivial S-acts, then all S-acts are (r,t)-injective.*

(2) *If $\mathbb{R}_r = \mathbb{S}_t$ or $\mathbb{R}_t = \mathbb{S}_r$, then all S-acts are (r,t)-injective.*

(3) *If $\mathbb{R}_r = \mathbb{R}_t$, then an S-act Q is (r,t)-injective if and only if Q is A-injective, for every $A \in \mathbb{R}_r$.*

Proof. To prove (1), we should note that if $\mathbb{R}_r \cap \mathbb{R}_t$ consists of trivial S-acts, then the only r-dense subact of a t-radical S-act A is A. Indeed, the closedness of $\mathbb{R}_t$ under homomorphic images implies that, for every r-dense subact B of A, $A/B \in \mathbb{R}_r \cap \mathbb{R}_t$. So A/B is a trivial S-act, by hypothesis. Therefore we have $A = B$ and the result follows from Theorem 3.8. Part (2) immediately follows from part (1). Also, part (3) is a corollary of Theorem 3.8. □

Remark 5.4. Given a Hoehnke radical r and a Kurosh-Amitsur radical t, if the radical classes $\mathbb{R}_r$ and $\mathbb{R}_t$ are closed under coproducts, then, for every $A \in \mathbb{R}_t$ and trivial S-act Θ, $A \coprod \Theta \in \mathbb{R}_t$ and $(A \coprod \Theta)/A \in \mathbb{R}_r$. So one can easily prove that

(1) every (r,t)-injective non-t-semisimple S-act contains a zero element.

(2) given a family $\{Q_i\}_{i\in I}$ of S-acts which are non-t-semisimple, $\prod_{i\in I} Q_i$, is (r,t)-injective if and only if each Q_i is (r,t)-injective.

Remark 5.5. Given two Hoehnke radicals r and t, if the coproduct of a family $\{Q_i\}_{i\in I}$ of S-acts, $\coprod_{i\in I} Q_i$ is (r,t)-injective then every Q_j with a zero element is (r,t)-injective.

Proof. Given an (r,t)-pair $(m : B \to A, f : B \to Q_j)$ in which Q_j has a zero element θ, f can be extended to

$$\begin{array}{rccl} \bar{f}: & A & \longrightarrow & Q_j \\ & a & \mapsto & \begin{cases} \hat{f}(a) & \hat{f}(a) \in Q_j \\ \theta & \text{Other waise.} \end{cases} \end{array}$$

through m, in which $\hat{f} : A \to \coprod_{i\in I} Q_i$ is the extention of $\iota_j \circ f$ through m, where $\iota_j : Q_j \to \coprod_{i\in I} Q_i$ is the injection map. □

Theorem 5.6. *Given two Hoehnke radicals r and t, if $\mathbb{S}_t$ is closed under coproducts, then the following statements are equivalent:*

(1) *The coproduct of (r,t)-injective S-acts is (r,t)-injective.*

(2) *$\Theta \coprod \Theta$ is (r,t)-injective, where Θ is a trivial S-act.*

Proof. (1) $\Rightarrow$ (2) is clearly true.

(2) $\Rightarrow$ (1) To prove, we show that, for every r-monomorphism $m : B \to A$ with $A \in \mathbb{R}_t$, each homomorphism $f : B \to \coprod_{i\in I} X_i$, in which each X_i is an (r,t)-injective S-act, can be extended to a homomorphism $\bar{f} : A \to \coprod_{i\in I} X_i$ such that $\bar{f}m = f$. To do so, we choose a fixed element j of I with $f(B) \cap X_j \neq \emptyset$ and define the homomorphism $f' : B \to \{\theta_1\} \amalg \{\theta_2\}$ by

$$f'(b) = \begin{cases} \theta_1 & f(b) \in X_j \\ \theta_2 & \text{otherwise.} \end{cases}$$

where $\{\theta_i\}$, $i = 1,2$, is the trivial S-act. So, using part (2), there exists a homomorphism $\bar{f}' : A \to \{\theta_1\} \amalg \{\theta_2\}$ such that $\bar{f}'m = f$. But $A \in \mathbb{R}_t$ and $\{\theta_1\} \amalg \{\theta_2\} \in \mathbb{S}_t$. Hence $\bar{f}'(A) = \{\theta_1\}$. Thus $f'(B) = \{\theta_1\}$. That is $f(B) \subseteq X_j$. So, the (r,t)-injectivity of X_j implies that there exists a homomorphism $\bar{f} : A \to \coprod_{i\in I} X_i$ such that $\bar{f}m = f$. □

Definition 5.7. By an (r,t)-*injective hull* of an S-act A, denoted by $E_{(r,t)}(A)$, we mean a minimal (r,t)-injective S-act containing A.

Theorem 5.8. *Let r be a hereditary Hoehnke radical and t be a pre-Kurosh radical. Then (r,t)-injective hull $E_{(r,t)}(A)$, for every S-act A, exists and it is determined uniquely up to isomorphism.*

Proof. First, we note that $R^t_{E_r(A)} \cup A$ is an (r,t)-injective hull of A. Indeed, $R^t_{E_r(A)} \cup A$ is an (r,t)-injective S-act, by Theorem 4.2. Also, if $A \leq P \leq R^t_{E_r(A)} \cup A$ is an (r,t)-injective S-act, then Theorem 4.5 implies that $R^t_P = R^t_{E_r(A)}$. So, $R^t_{E_r(A)} \cup A \leq P$. Therefore $P = R^t_{E_r(A)} \cup A$.

□

Definition 5.9. Given a radical r over **S-Act** and an S-act A, we define a closure operator c_r of the category of **S-Act** by $c^A_r(B) := B \cup R^r_A$ for any subact B of A.

Lemma 5.10. *Given a Hoehnke radical r, the closure operator c_r is both idempotent and weakly hereditary.*

Proof. The closure operator c_r is idempotent, because for each subact B of an S-act A, we have

$$c^A_r(c^A_r(B)) = c^A_r(B) \cup R^r_A = B \cup R^r_A \cup R^r_A = B \cup R^r_A = c^A_r(B).$$

Also the closure operator c_r is weakly hereditary, because for each subact B of an S-act A, we have

$$c_r^{c^A_r(B)}(B) = B \cup R^r_{c^A_r(B)} = B \cup R^r_{B\cup R^r_A} = B \cup R^r_B \cup R^r_A = B \cup R^r_A = c^A_r(B).$$

□

Definition 5.11. A subact B of an S-act A is said to be c_r*-large* if B is both large and c_r-dense in A. In this case, we call A to be a c_r*-essential extension* of B. Also, a c_r-monomorphism $\iota : B \to A$ is called c_r*-essential monomorphism* if $\iota(B)$ is c_r-large in A.

Definition 5.12. Let r and t be two Hoehnke radicals.

(i) An S-act Q is said to be c_r*-injective* if every homomorphism $f : B \longrightarrow Q$ with nonempty domain can be extended to a homomorphism $\bar{f} : A \longrightarrow Q$ through every c_r-monomorphism $i : B \to A$, that is, $f = \bar{f}i$. Also an S-act, denoted by $E_{c_r}(A)$, is called c_r*-injective hull of an S-act A* whenever it is c_r-injective and a c_r-essential extension of A.

(ii) An S-act Q is called (r, c_t)*-injective* if every homomorphism $f : B \longrightarrow Q$ with nonempty domain can be extended to a homomorphism $\bar{f} : A \longrightarrow Q$ through every homomorphism $i : B \longrightarrow A$ which is both r-monomorphism and c_t-monomorphism. Also an S-act, denoted by $E_{c_{(r,c_t)}}(A)$, is called (r, c_t)*-injective hull of an S-act A* whenever it is (r, c_t)-injective and an (r, c_t)-essential extension of A.

Proposition 5.13. *Let r be a hereditary Hoehnke radical and t be a Hoehnke radical. Then, for each S-act A, $E_{c_r}(A) = A \cup R^r_{E(A)}$ and $E_{(r,c_t)}(A) = A \cup R^t_{E_r(A)}$.*

Proof. Obviously, the class of essential monomorphisms is left cancellable for monomorphisms. Also since r is hereditary, the class of r-essential monomorphisms is left cancellable for monomorphisms, and, by Lemma 5.10, c_r is idempotent and weakly-hereditary. So, by Proposition 2.6, we have $E_{c_r}(A) = c_r^{E(A)}(A) = A \cup R^r_{E(A)}$ and $E_{(r,c_t)}(A) = c_t^{E_r(A)}(A) = A \cup R^r_{E_r(A)}$. □

Remark 5.14. Given a Hoehnke radical t and a hereditary Hoehnke radical r, the fact that the notion of (r, c_t)-injectivity is well-behavior follows from Proposition 5.13 and Corollaries 2.7 and 2.8.

Definition 5.15. Given two Hoehnke radicals r and t, and an S-act A,

(i) we say that an S-act X is an (r, t)*-essential extension* of A whenever X is an essential extension of A with $X \leq R^t_{E_r(A)} \cup A$.

(ii) We say that A is (r, t)*-absolute retract* if every r-monomorphism $m : X \to B$ with $B \in \mathbb{R}_t$ and $X \in \Sigma_{t(A)}$ is a retraction.

Proposition 5.16. *Let r be a hereditary Hoehnke radical and t be a pre-Kurosh radical. Then the following conditions are equivalent, for an S-act Q:*

(1) *Q is (r,t)-injective.*

(2) *Q has no proper (r,t)-essential extension.*

(3) *Q is (r,t)-absolute retract.*

(4) *Q is (r,c_t)-injective.*

(5) *Q has no proper (r,c_t)-essential extension.*

(6) *Q is (r,c_t)-absolute retract.*

Proof. (1) $\Rightarrow$ (2) For a given (r,t)-essential extension A of Q, we have $Q \leq A \leq R^t_{E_r(Q)} \cup Q$. But, since Q is (r,t)-injective, Theorem 4.5 implies $R^t_{E_r(Q)} \cup Q = R^t_Q \cup Q \subseteq Q$, and hence $R^t_{E_r(Q)} \cup Q = Q$. Therefore $Q = A$.

(2) $\Rightarrow$ (1) By Definition 5.15, $R^t_{E_r(Q)} \cup Q$ is an (r,t)-essential extension of Q. So, by hypothesis, $Q = R^t_{E_r(Q)} \cup Q$. But, since, by Theorem 5.8, $R^t_{E_r(Q)} \cup Q$ is (r,t)-injective hull of Q, Q is (r,t)-injective.

(1) $\Rightarrow$ (3) By Theorem 5.2, every $X \in \Sigma_{t(Q)}$ is (r,t)-injective. So, for every r-monomorphism $m : X \to B$ with $B \in \mathbb{R}_t$ and $X \in \Sigma_{t(Q)}$, $\mathtt{id}_X$ can be extended to $\overline{\mathtt{id}_X} : B \to X$ through m. That is m is a retraction.

(3) $\Rightarrow$ (1) First we note that for every $X \in \Sigma_{t(Q)}$, there exists $\bar{X} \in \Sigma_{t(E_r(Q))}$ such that $X \leq \bar{X}$. Also, heredity of r indicates that X is r-dense in $\bar{X} \in \mathbb{R}_t$. So, the inclusion map $X \hookrightarrow \bar{X}$ is both monomorphism and retraction. Hence $X = \bar{X}$. Therefore, by Theorem 4.5, Q is (r,t)-injective.

The implications (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) follow from Corollary 2.8 and Lemma 5.10 and the implications (1) $\Leftrightarrow$ (4) follows from the fact that $E_{(r,c_t)}(Q) = E_{(r,t)}(Q)$, for every $Q \in$ **S-Act**, see Proposition 5.13 and Theorem 5.8. $\square$

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