



Representation of H -closed monoreflections in archimedean ℓ -groups with weak unit

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Communicated by Themba Dube

Abstract. The category of the title is called \mathcal{W} . This has all free objects $F(I)$ (I a set). For an object class \mathcal{A} , $H\mathcal{A}$ consists of all homomorphic images of \mathcal{A} -objects. This note continues the study of the H -closed monoreflections (\mathcal{R}, r) (meaning $H\mathcal{R} = \mathcal{R}$), about which we show (*inter alia*): $A \in \mathcal{A}$ if and only if A is a countably up-directed union from $H\{rF(\omega)\}$. The meaning of this is then analyzed for two important cases: the maximum essential monoreflection $r = c^3$, where $c^3F(\omega) = C(\mathbb{R}^\omega)$, and $C \in H\{c(\mathbb{R}^\omega)\}$ means $C = C(T)$, for T a closed subspace of \mathbb{R}^ω ; the epicomplete, and maximum, monoreflection, $r = \beta$, where $\beta F(\omega) = B(\mathbb{R}^\omega)$, the Baire functions, and $E \in H\{B(\mathbb{R}^\omega)\}$ means E is an epicompletion (not “the”) of such a $C(T)$.

1 Introduction

\mathcal{W} is the category of archimedean ℓ -groups G with distinguished weak order unit e_G , and morphisms $G \xrightarrow{\varphi} H$ the ℓ -group homomorphisms with $\varphi(e_G) =$

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Keywords: Archimedean ℓ -group, H -closed monoreflection, Yosida representation, countable composition, epicomplete, Baire functions.

Mathematics Subject Classification[2010]: 06F20, 18A40, 46A40, 54F15.

Received: 22 July 2017, Accepted: 26 December 2017

ISSN Print: 2345-5853 Online: 2345-5861

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e_H . We compress the discussion in §1 of [11], which see for more detail. “ $A \leq B$ ” means A is a \mathcal{W} -subobject of B .

The forgetful functor $\mathcal{W} \rightarrow \text{Sets}$ has the left adjoint F . An $F(I)$ is the free object on the set I , and this is the \mathcal{W} -subobject of $\mathbb{R}^{\mathbb{R}^I}$ generated by the constant function 1, and all projections $\pi_i : \mathbb{R}^I \rightarrow \mathbb{R}$ ($i \mapsto \pi_i$ is the “insertion of generators” $I \hookrightarrow F(I)$).

A full subcategory \mathcal{R} of \mathcal{W} is monoreflective if $\forall A \in \mathcal{W} \exists$ monic $A \xrightarrow{r_A} rA$, $rA \in \mathcal{R}$, with the property: $\forall A \xrightarrow{\varphi} R$, $R \in \mathcal{R}$, $\exists!$ $rA \xrightarrow{\bar{\varphi}} R$ with $\bar{\varphi}r_A = \varphi$. We usually write $A \leq rA$ for the r_A . We abuse language and notation by saying as convenient (\mathcal{R}, r) or \mathcal{R} , or r , is a monoreflection.

The class of monoreflections is ordered by: $r \leq s$ means $\forall A \exists$ monic f with $s_A = fr_A$.

Let $M \xrightarrow{m} M' \in \mathcal{W}$. Then, $A \in \text{inj}\{m\}$ means: $\forall M \xrightarrow{\varphi} A \exists M' \xrightarrow{\varphi'} A$ with $\varphi'm = \varphi$.

“ ω ” stands for the natural numbers, or any countable set, or the ordinal or cardinal.

Theorem 1.1 ([11], 2.7). *Suppose (\mathcal{R}, r) is an H -closed monoreflection. Then $\mathcal{R} = \text{inj}\{F(\omega) \leq rF(\omega)\}$.*

Theorem 1.1 is one of the main results of [11] and is the cornerstone of that paper. It devolves from categorical generalities, and many special features of \mathcal{W} , some of which we describe below, and some later when needed.

Another main result of [11] is the characterization of the $rF(\omega)$ in Theorem 1.1. Namely, 3.6 there says these are exactly the S with $F(\omega) \stackrel{\sigma}{\leq} S \leq B(\mathbb{R}^\omega)$ (B the Baire functions), with σ epic and $S \circ S^\omega = S$ (that is, $\forall s$ and countable $\{s_n\}$ from S , the function $\mathbb{R}^\omega \xrightarrow{\langle s_n \rangle} \mathbb{R}^\omega \xrightarrow{s} \mathbb{R}$ lies in S). The cases for c^3 and β are mentioned in the Abstract, and will be deployed below.

Let \bigcup^ω denote a countably up-directed union, in Sets or in \mathcal{W} . For $\mathcal{A} \subseteq \text{Sets}$ or \mathcal{W} , $A \in \bigcup^\omega \mathcal{A}$ means there is a family \mathcal{A}' of \mathcal{A} -subobjects of A with $A = \bigcup^\omega \mathcal{A}'$.

For $I \in \text{Sets}$, let $\mathcal{P}_0(I) = \{J \subseteq I \mid |J| \leq \omega\}$. Then $I = \bigcup^\omega \mathcal{P}_0(I)$. For $A \in \mathcal{W}$, $A = \bigcup^\omega \{B \leq A \mid |B| \leq \omega\}$. From the form of the $F(I)$, and the fact that any $f \in C(\mathbb{R}^I)$ factors through a countable subproduct, we have $F(I) = \bigcup^\omega \{F(J) \mid J \in \mathcal{P}_0(I)\}$.

A crucial ingredient to what we have said so far, and necessary later, is the Yosida representation of \mathcal{W} -objects:

\mathbb{R} is the real numbers, and $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ under the obvious topology and order. For X a topological space, $D(X) = \{f \in C(X, \overline{\mathbb{R}}) \mid f^{-1}\mathbb{R} \text{ dense in } X\}$. This is a lattice containing $C(X)$, but has only partly defined $+$. For $A \in \mathcal{W}$, $A \leq D(X)$ means $A \overset{\mathcal{W}}{\approx} A' \subseteq D(X)$, where A' is closed under the partly defined data required to make $A' \in \mathcal{W}$.

The Yosida representation of $A \in \mathcal{W}$ (see [12]) says:

- (1) $A \leq D(\mathcal{Y}A)$ for a unique compact Hausdorff $\mathcal{Y}A$ for which A separates the points.
- (2) For $A \xrightarrow{\varphi} B \in \mathcal{W}$, there is a unique continuous $\mathcal{Y}A \xleftarrow{\mathcal{Y}\varphi} \mathcal{Y}B$ for which $\varphi(a) = a \circ \mathcal{Y}\varphi \ \forall a \in A$. If φ is onto, then $\mathcal{Y}\varphi$ is an embedding, $\mathcal{Y}A \hookrightarrow \mathcal{Y}B$.

The Yosida representation of $C(X)$, X Tychonoff, is Čech-Stone extension $C(X) \ni f \mapsto \beta f \in D(\beta X)$.

2 Main Theorem

We expand on Theorem 1.1.

Theorem 2.1. *Suppose (\mathcal{R}, r) is an H -closed monoreflection in \mathcal{W} . For $A \in \mathcal{W}$, the following are equivalent:*

- (1) $A \in \mathcal{R}$.
- (2) There is I with a surjection $rF(I) \twoheadrightarrow A$.
- (3) $A \in \text{inj}\{F(\omega) \leq rF(\omega)\}$.
- (4) Each countable $B \overset{i_B}{\leq} A$ (i_B labels the inclusion) has the property

$$\text{there is } rB \xrightarrow{\bar{i}_B} A \text{ with } \bar{i}_B rB = i_B. \quad (*)$$

- (5) $A \in \overset{\omega}{\bigcup} H\{rF(\omega)\}$.
- (6) $A \in \overset{\omega}{\bigcup} \mathcal{R}$.

Proof. (1) \Leftrightarrow (2) is quite general: For (1) \implies (2), take $F(I) \xrightarrow{\varphi} A$. We have $rF(I) \xrightarrow{\bar{\varphi}} A$ with $\bar{\varphi} r_{F(I)} = \varphi$ because $A \in \mathcal{R}$, and $\bar{\varphi}$ is a surjection.

(2) \implies (1) because $\mathcal{R} = H\mathcal{R}$.

(1) \Leftrightarrow (3) is exactly Theorem 1.1.

(1) \implies (2) is obvious (in fact, for any $B \leq A$).

(4) \implies (5). We isolate two steps of the proof, just assuming (\mathcal{R}, r) monoreflective (not assuming $(H\mathcal{R} = \mathcal{R})$. Proofs of these items are obvious.

Step (i). Suppose $B \in \mathcal{W}$ and $|B| \leq \omega$. Take any $F(\omega) \xrightarrow{\varphi} B$. We then take $\bar{\varphi}$ as shown

$$\begin{array}{ccc} F(\omega) & \leq & rF(\omega) \\ \varphi \downarrow & & \downarrow \bar{\varphi} \\ B & \leq & rB \end{array}$$

commuting, so $B \leq \bar{\varphi}(rF(\omega)) \leq rB$.

Step (ii). Suppose $A = \bigcup_{\alpha}^{\omega} B_{\alpha}$, where each $B_{\alpha} \leq A$ has the property (*) in (4), with corresponding $\bar{i}_{B_{\alpha}}$. Then $A = \bigcup_{\alpha}^{\omega} \bar{i}_{B_{\alpha}}(rB_{\alpha})$.

Now suppose $H\mathcal{R} = \mathcal{R}$. In Step (i), we then have $\bar{\varphi}_R(rF(\omega)) \in \mathcal{R}$, thus $\bar{\varphi}_R(rF(\omega)) = rB$, because the embedding $B \leq rB$ is “minimal to \mathcal{R} ” (see [10]). This makes $rB \in H\{rF(\omega)\}$.

Finally: Write $A = \bigcup_{\alpha}^{\omega} \{B \mid B \leq A, |B| \leq \omega\}$. By (4), step (ii) applies and $A = \bigcup_{\alpha}^{\omega} \{\bar{i}_B(rB) \mid B \leq A, |B| \leq \omega\}$. Since each $rB \in H\{rF(\omega)\}$, also each $\bar{i}_B(rB) \in H\{rF(\omega)\}$. Thus, (5).

(5) \implies (6) because $H\mathcal{R} = \mathcal{R}$.

(6) \implies (3) This amounts to showing that $\text{inj}\{F(\omega) \leq rF(\omega)\}$ is closed under $\bigcup_{\alpha}^{\omega}$, since we already noted (3) \Leftrightarrow (1). So suppose $A = \bigcup_{\alpha}^{\omega} R_{\alpha}$, $R_{\alpha} = \text{inj}\{F(\omega) \leq rF(\omega)\}$, and take $F(\omega) \xrightarrow{\varphi} A$. Since $|F(\omega)| = \omega$, also $|\varphi(F(\omega))| \leq \omega$, and $\varphi(F(\omega)) \leq$ some R_{α} . So there is $rF(\omega) \xrightarrow{\bar{\varphi}} R_{\alpha} \in A$ extending φ . \square

We now examine 2.1 for the important cases $r = c^3$ and $r = \beta$.

3 c^3 (Closed under countable composition)

“ c^3 ” stands for “closed under countable composition”, originally studied in [13]. The definition goes as follows.

Each $A \in \mathcal{W}$ has its Yosida representation $A \leq D(\mathcal{Y}A)$. A sequence a_1, a_2, \dots from A has $\bigcap a^{-1}\mathbb{R}$ dense in $\mathcal{Y}A$ (Baire Category Theorem) and

let $\langle a_n \rangle = \bigcap_n a_n^{-1}\mathbb{R} \rightarrow \mathbb{R}^\omega$ be the function defined by $\pi_j(\langle a_n \rangle(x)) = a_j(x)$ $\forall j$. For $f \in C(\mathbb{R}^\omega)$, we have the composition $\bigcap_m a_n^{-1}(\mathbb{R}) \xrightarrow{\langle a_n \rangle} \mathbb{R}^\omega \xrightarrow{f} \mathbb{R}$. A is c^3 if each such $f \circ \langle a_n \rangle$ extends over $\mathcal{Y}A$ to an element of A .

c^3 will denote either the object class, or reflections $A \leq c^3 A$. We assemble known facts.

Theorem 3.1. (Each item without specific reference can be located in [11] §1, with reference to original sources.)

(a) ([13]). A is c^3 if and only if $A = \overset{\omega}{\bigcup}\{C(\bigcap_n a_n^{-1}\mathbb{R}) \mid a_1, a_2, \dots \in A\}$ if and only if there is a Tychonoff space X and a surjection $C(X) \rightarrow A$.

(b) A is c^3 if and only if $A \approx C(\mathcal{L})$, \mathcal{L} a locale (aka, the f -ring of real-valued continuous functions on a frame \mathcal{L}).

(c) c^3 is monoreflective, with reflections $A \leq c^3 A = \varinjlim\{C(\bigcap_n a_n^{-1}\mathbb{R}) \mid a_1, a_2, \dots \in A\}$, and is an essential monoreflection (meaning that $A \leq c^3 A$ is a essential monic).

The class c^3 is H -closed.

(d) c^3 is the largest essential monoreflection (with the smallest class of objects).

(e) \forall set I , $c^3 F(I) = C(\mathbb{R}^I) = \overset{\omega}{\bigcup}\{C(\mathbb{R}^J) \mid J \in \mathcal{P}_0(I)\}$.

We consider the meaning of 2.1 (4) for $r = c^3$.

A Tychonoff space X is called Čech-complete if X is G_δ in βX (see [7]).

We abbreviate “Lindelöf and Čech-complete” to “ $L\check{C}$ ”.

Theorem 3.2. If X is $L\check{C}$, then $H\{C(X)\} = \{C(T) \mid T \text{ closed in } X\}$.

Proof. First note: For any Tychonoff space X and $T \subseteq X$, the restriction $C(X) \ni f \mapsto f|_T \in C(T)$ defines a \mathcal{W} -homomorphism $C(X) \xrightarrow{\rho_T} C(T)$, and ρ_T is onto if and only if T is C -embedded in X (which entails the closure \overline{T} is C -embedded.) (See [8].)

Now suppose X is $L\check{C}$. Then X is normal, so any closed T is C -embedded, thus $C(T) \in H\{C(X)\}$.

For the converse, we shall use details of the Yosida representation; see §1. Any $A \xrightarrow{\varphi} B$ has the quasi-dual embedding $\mathcal{Y}A \leftrightarrow \mathcal{Y}B$ for which $\varphi(a) = a|_{\mathcal{Y}B} \forall a \in A$. This entails $a^{-1}\mathbb{R} \cap \mathcal{Y}B$ dense in $\mathcal{Y}B$, and thus $\forall a_1, a_2, \dots \in A$, $(\bigcap_n a_n^{-1}(\mathbb{R})) \cap \mathcal{Y}B = \bigcap_n (a_n^{-1}\mathbb{R} \cap \mathcal{Y}B)$ dense in $\mathcal{Y}B$ (Baire Category Theorem).

Now, the Yosida representation of a $C(X)$ is extension over Čech-Stone compactification βX , as $C(X) \approx \{\beta a \mid a \in C(X)\}$. And X is $\check{L}\check{C}$ if and only if $\exists a_1, a_2, \dots \in C(X)$ with $X = \bigcap_n (\beta a_n)^{-1}\mathbb{R}$.

Suppose X is $\check{L}\check{C}$, and $C(X) \xrightarrow{\varphi} B$ with Yosida dual embedding $\beta X \leftarrow \mathcal{Y}B$. Take $\{a_n\} \subseteq C(X)$ with $X = \bigcap_n (\beta a_n)^{-1}\mathbb{R}$ as above. Then $T = X \cap \mathcal{Y}B = \bigcap_n (\beta a_n)^{-1}\mathbb{R} \cap \mathcal{Y}B$ is dense in $\mathcal{Y}B$. (So we can view $B \leq C(T)$), and closed in the normal X (thus C -embedded) so $B = C(T)$. \square

Summing up, we interpret Theorem 2.1 for c^3 through Theorem 3.2 and some of Theorem 3.1.

Corollary 3.3. *For $A \in \mathcal{W}$, the following are equivalent:*

- (1) $A \in c^3$.
- (2) *There is I with a surjection $C(\mathbb{R}^I) \twoheadrightarrow A$.*
- (3) $A \in \text{inj} \{F(\omega) \leq C(\mathbb{R}^\omega)\}$.
- (4) *For any countable $B \leq A$, also $c^3 B \leq A$.*
- (5) $A \subseteq \bigcup_{\omega} \{C(T) \mid T \text{ closed in } \mathbb{R}^\omega\}$.
- (6) $A \in \uparrow c^3$.

Proof. This is all quite immediate. We just note: (4) is just Theorem 2.1(5), using Theorem 3.2 for $X = \mathbb{R}^\omega$.

(4) is the statement that in Theorem 2.1(4) the \bar{i}_B are one-to-one. This follows solely from the essentiality of the reflection maps $B \leq c^3 B$. \square

Remark 3.4. (a) $A \in \uparrow c^3 \not\Rightarrow A \in \bigcup_{\omega} c^3$. An example is $A = \{f \in C(\mathbb{R}^\omega) \mid \exists \text{ finite } F \subseteq \omega \text{ s.t. } f = \bar{f} \circ \pi_F\}$.

(b) Corollary 3.3 (3) and (5) are to be compared with Theorem 3.1(a). The X in Theorem 3.1(a) is $\mathcal{Y}A \times \mathbb{N}$.

(c) We note [7], p. 74: T is (\approx) a closed subspace of \mathbb{R}^ω if and only if T is completely metrizable and separable.

(d) In Corollary 3.3 (5), the $A = \bigcup_{\omega} C(T)$'s is a countably directed direct limit, $A = \varinjlim_{\omega} C(T)$'s. The Yosida functor converts this to an inverse limit $\mathcal{Y}A = \varprojlim_{\omega} \beta T$'s. Using $A = C(X)$ with X real compact, and a little fiddling yields $\bar{X} = \varprojlim_{\omega} T$'s, and if X is compact, so are the T 's. This is more or less a result of Pásynkov [15]. See also [7], p. 220.

(e) An essential reflection (\mathcal{R}, r) has $r \leq c^3$ (Theorem 3.1 (d)), and if $\mathcal{R} = H\mathcal{R}$, Corollary 3.3 holds *mutatis mutandis*. For $rF(\omega) = S$ (see

the second paragraph after Theorem 1.1), we have $F(\omega) \stackrel{\sigma}{\leq} S \leq C(\mathbb{R}^\omega)$, and “ σ epic” is automatic. Examples of this are: \mathcal{R} = “rings” (\mathcal{W} -objects A with a compatible f -ring multiplication with identity the \mathcal{W} -unit e_A), vector lattices, algebras, For example: for rings, $rF(\omega)$ is the sub- f -ring of $C(\mathbb{R}^\omega)$ generated by $F(\omega)$. In Corollary 3.3 (4), each $C(T)$ is to be replaced by the set of restrictions $rF(\omega)|T$. An additional feature of any essential r is that $rF(\omega)|T = r(F(\omega)|T)$.

(f) The present paper began with an analysis of a version of Corollary 3.3 and some related matters, in the view of a c^3 -object as the f -ring of real-valued continuous functions on a frame. As such, it was reported in [6]: where c^3 was taken as condition 3.3(3), thus avoiding a reference to the Yosida representation and the reflection is then given an explicit frame-theoretic form. See [4] for details.

4 β (Epicomplete)

E is called epicomplete if $E \xrightarrow{\varphi} \bullet$ monic and epic implies φ an isomorphism. The class of epicomplete objects is denoted EC .

Recall that, for a Tychonoff space X , $B(X)$ denotes the \mathcal{W} -object of real-valued Baire functions on X .

We summarize known features of EC , prior to the interpretation of Theorem 2.1 for $\mathcal{R} = EC$.

Theorem 4.1. (Each item without specific reference can be located in [11], with reference to original sources.)

(a) $E \in EC$ if and only if E is σ -complete both conditionally, and laterally if and only if $E \approx D(X)$ with X basically disconnected (the X is $\mathcal{Y}E$). Thus, any $B(X) \in EC$.

(b) ([3]). $E \approx C(\mathcal{P})$ with \mathcal{P} a P -locale. (Such a \mathcal{P} is the localic intersection of $\{S \mid S \text{ is dense cozero in } \mathcal{Y}E\}$.)

(c) EC is monoreflective, thus the maximum monoreflection. The reflection of A is $\beta A = B(\mathcal{Y}A)/N$, for a certain σ -ideal N .

EC is H -closed, thus $EC = H\{B(K) \mid K \text{ compact}\}$.

(d) If X is Lindelöf and Čech-complete, then $\beta C(X) = B(X)$.

(e) For every set I , $\beta F(I) = B(\mathbb{R}^I) = \bigcup_{\omega} \{B(\mathbb{R}^J) \mid J \in \mathcal{P}_0(I)\}$.

We now interpret Theorem 2.1. Most of this is the routine writing-down of items in Theorem 2.1 using information in Theorem 4.1. An exception is Theorem 2.1 (5), which says $A \in H\{B(\mathbb{R}^\omega)\}$. “An” epicompletion of $A \in \mathcal{W}$ is an epic $A \leq E$, with $E \in EC$. These are exactly the quotients over A of βA .

Theorem 4.2. *Suppose X is $L\check{C}$ (as is \mathbb{R}^ω).*

(a) $E \in H\{B(X)\}$ if and only if there is F closed in X such that E is an epicompletion of $C(F)$.

(b) (Note that an F in (a) is again $L\check{C}$.) $C(X)$ has a unique epicompletion if and only if X is discrete and countable (and thus $X \approx \mathbb{N}$, $C(X) \approx C(\mathbb{N})$, is already EC).

(c) If X is not countable discrete, there are many epicompletions of $C(X)$.

Proof. (a) Suppose $E \in H\{B(X)\}$, as $B(X) \xrightarrow{\varphi} E$. We have

$$\begin{array}{ccc} C(X) & \xrightarrow{\beta_C} & \beta C(X) = B(X) \quad (\text{by Theorem 4.1(d)}) \\ \varphi_0 \downarrow & & \downarrow \varphi \\ \varphi(C(X)) & \xrightarrow{e} & E \end{array}$$

where φ_0 is the restriction of φ , e labels the inclusion, and $\varphi\beta_C = e\varphi_0$ (obviously), so e is epic (as a second factor of the epic $\varphi\beta_C$).

By Theorem 3.2, $\varphi(C(X))$ is the desired $C(F)$.

Suppose F is closed in X and $C(F) \xrightarrow{e} E$ is an epicompletion. We then have

$$\begin{array}{ccc} C(X) & \xrightarrow{\beta_C} & \beta C(X) \\ \rho \downarrow & & \downarrow \bar{\rho} \\ C(F) & \xrightarrow{e} & E \end{array}$$

where ρ is the restriction map described at the beginning of the proof of Theorem 3.2, and then $\exists \bar{\rho}$ with $\bar{\rho}\beta_C = e\rho$ by the universal mapping property of β .

We have $C(F) \xrightarrow{i} \bar{\rho}(\beta C(X)) \xrightarrow{j} E$ (i, j are labels) with $ji = e$. But $\bar{\rho}(\beta C(X)) \in EC$ (by Theorem 4.1(c)), and e is epic, thus also j . So j is equality.

(b) If $C(X) \approx C(\mathbb{N})$, already $C(X) \in EC$, so is its only epicompletion.

If $C(X)$ has a unique epicompletion, it must be $C(X) \leq B(X)$ (Theorem 4.1 (d)), and this must be an essential embedding (because any $A \in \mathcal{W}$ has a (unique) essential epicompletion ([2], §9)). If X has a non-void nowhere dense zero-set Z , then the characteristic function $\chi(Z) \in B(X)$, and there is no $0 < a \in C(X)$ with $a \leq \chi(Z)$: $C(X) \leq B(X)$ is not essential. Thus there is no such Z , so X is what is called an almost P -space. But the only almost P -space which is $\check{L}\check{C}$ is (\approx) \mathbb{N} .

(c) See [1] and [2] for several constructions. We omit details. \square

Referring to Theorem 4.2, let $\mathcal{ECS}(\mathbb{R}^\omega)$ stand for the family of epicompletions of objects of the form $C(T)$, for T closed in \mathbb{R}^ω .

Summing up, we write down Theorem 2.1 for $\mathcal{W} \xrightarrow{\beta} EC$ using Theorem 4.2 and some of Theorem 4.1.

Corollary 4.3. *For $A \in \mathcal{W}$, the following are equivalent:*

- (1) $A \in EC$.
- (2) There is I with a surjection $B(\mathbb{R}^I) \twoheadrightarrow A$.
- (3) $A \in \text{inj}\{F(\omega) \leq B(\mathbb{R}^\omega)\}$.
- (4) Each countable $B \stackrel{i_B}{\leq} A$, has the property

$$\text{there is } \beta B \xrightarrow{\bar{i}_B} A \text{ with } \bar{i}_B \beta B = i_B. \quad (*)$$

- (5) $A \in \bigcup_{\omega} \mathcal{ECS}(\mathbb{R}^\omega)$.
- (6) $A \in \bigcup_{\omega} EC$.

The comparison of Corollary 4.3 (4) and (5) with Corollary 3.3 (4) and (5), shows a huge difference between c^3 (or any essential reflection) with β and identifies some special classes of EC objects which might deserve further study. (It is quite rare that any $A \leq \beta A$ is essential; see [2], §9.)

We consider the analogue of Corollary 3.3 (4) for β . Recall that for $B \leq A$, $\beta B \leq A$ means that the \bar{i}_B in 2.8 (4) is one-to-one.

Theorem 4.4. *Suppose $A \in \mathcal{W}$. For every countable $B \leq A$,*

$$\beta B \leq A \text{ if and only if } A \approx \mathbb{R}^n \text{ for some } n \in \mathbb{N}. \quad (*)$$

Proof. Notice that either the condition implies $A \in EC$: for $A \in EC$ (or just a vector lattice), $A \approx \mathbb{R}^n$ ($n \in \mathbb{N}$) means $|\mathcal{Y}A| = n$ (and $\mathbb{R}^n = C(\{0, 1, \dots, n-1\})$).

(\Leftarrow) We omit the easy proof.

(\Rightarrow) We show that $\mathcal{Y}A$ infinite $\Rightarrow A$ fails (*).

(i) $A = C(\mathbb{N})$ fails (*).

(ii) If $A \in EC$ and $\mathcal{Y}A$ is infinite, then there is an embedding $C(\mathbb{N}) \leq A$.

(iii) If $A \in EC$ and $\mathcal{Y}A$ is infinite, then A fails (*).

For (i): Let $B \leq C(\mathbb{N})$ be generated by rational multiples of the characteristic functions χ_p of the $p \in \mathbb{N}$. A little thought reveals that the uniform completion $uB = c^3B = C(\alpha\mathbb{N})$, where $\alpha\mathbb{N} = \mathbb{N} \cup \{\alpha\}$ the one-point compactification of \mathbb{N} . Then $\beta C(\alpha\mathbb{N}) = B(\alpha\mathbb{N})$ (Theorem 4.1 (d)). Then, the \bar{i}_B is not one-to-one: $\bar{i}_B(\psi_\alpha) = 0$.

For (ii): As with any infinite Hausdorff space, there is countable $L = \{x_n\} \subseteq \mathcal{Y}A$ on pairwise disjoint open sets $\{U_n\}$ in $\mathcal{Y}A$ with $U_n \cap L = \{x_n\} \forall n$. We have $L \approx \mathbb{N}$. Since $\mathcal{Y}A$ is basically disconnected, thus zero-dimensional ([8]). The U_n may be chosen clopen, and $\bar{U} \approx \beta\bar{U}$ (Čech-Stone). Choose any $p_0 \in U$, and retract $\mathcal{Y}A \xrightarrow{\rho} \bar{U}$ as $\rho(x) = [x, \text{ if } x \in \bar{U}; p_0 \text{ if } x \notin \bar{U}]$. Since \bar{U} is clopen, ρ is continuous.

Let $f \in C(L)$. Extend to $\bar{f} \in C(U)$ by $f(U_n) = \{f(x_n)\}$. Then extend \bar{f} to $\bar{\bar{f}} \in D(\beta\bar{U})$ (since $\bar{U} = \beta U$). Now $\bar{\bar{f}} \circ \rho \in D(\mathcal{Y}A)$ ($(\bar{\bar{f}} \circ \rho)^{-1}\mathbb{R} = \bar{f}^{-1}(\mathbb{R}) \subseteq \bar{U} - u$, which is nowhere dense). Define $C(L) \xrightarrow{\tilde{\rho}} D(\mathcal{Y}A) \in A$ as $\tilde{\rho}(f) = \bar{\bar{f}} \circ \rho$. Such compositions preserve ℓ -group operations (and multiplication) and constants, so $\tilde{\rho}(1) = 1$, and $\tilde{\rho} \in \mathcal{W}$.

For (iii): In (ii) we have $C(L) \xrightarrow{\tilde{\rho}} A$, which we re-name $C(\mathbb{N}) \xrightarrow{k} A$. In (i), we have countable $B \xrightarrow{i_B} C(\mathbb{N})$ with \bar{i}_B not 1-1. We have

$$\begin{array}{ccccc}
 B & & \xrightarrow{i_B} & c(\mathbb{N}) & \xrightarrow{k} & A \\
 \beta_B \downarrow & & \nearrow \bar{i}_B & & \nearrow \bar{j}_B & \\
 \beta B & & & & &
 \end{array}$$

with the inclusion $B \stackrel{j_B}{\leq} A$ being $j_B = ki_B$ and with $\bar{i}_B\beta_B = i_B$, $\bar{j}_B\beta_B = j_B = ki_B$. Thus $\bar{j}_B\beta_B = k\bar{i}_B\beta_B$, so $\bar{j}_B = k\bar{i}_B$ since β_B is epic. Since \bar{i}_B is not one-to-one, neither is \bar{j}_B . \square

Let $BS(\mathbb{R}^\omega) \equiv \{B(T) \mid T \text{ dense in } \mathbb{R}^\omega\}$. The analogue of Corollary 3.3(5) for β is the condition

$$A \in \bigcup_{\omega} \{B(T) \mid T \text{ closed in } \mathbb{R}^\omega\}. \quad (**)$$

All we have to say is: sometimes this happens, sometimes not.

Remark 4.5. (a) There are A satisfying (**): Obviously, any $B(T)$; less trivially, ([11]) for uncountable I , $B(\mathbb{R}^I) = \bigcup_{\omega} \{B(\mathbb{R}^J) \mid J \in \mathcal{P}_0(I)\}$.

(b) There are many A failing (**). The countable chain condition, ccc, of a space or \mathcal{W} -object is relevant here. X (resp., A) has ccc if there is no uncountable pairwise disjoint family of non-void open sets in X (respectively, non-zero positive elements in A). A has ccc if and only if $\mathcal{Y}A$ does (because $\text{coz}A$ is a base in $\mathcal{Y}A$).

If A has ccc and satisfies (**), then in the Yosida representation $A \approx D(\mathcal{Y}A)$, each $a \in A$ is locally constant on a dense open subset of $\mathcal{Y}A$. (If $A = \bigcup_{\omega} B(T_\alpha)$, then each $B(T_\alpha)$ has ccc, and it follows that T_α is a copy \mathbb{N}_α of \mathbb{N} . For each α , $C(\mathbb{N}_\alpha) = \beta\mathbb{N}_\alpha \stackrel{\tilde{c}}{\leftarrow} \mathcal{Y}A$). If $a \in C(\mathbb{N}_\alpha)$, then $a \text{ " = " } a \circ \tau$ is locally constant on $\tau^{-1}(\mathbb{N}_\alpha)$.)

Consider the absolute (projective cover) $[0, 1] \stackrel{\pi}{\leftarrow} a[0, 1]$. Using irreducibility of π : Since $[0, 1]$ has ccc, so do $a[0, 1]$, and also $A = D(a[0, 1])$. Here $C([0, 1]) \leq A$, as $f \mapsto f \circ \pi$. $\underline{\mathbb{N}}_0$ continuous nonconstant f has $f \circ \pi$ locally constant on a dense subset of $a[0, 1]$. Thus A fails (**).

(c) The class EC consists exactly of the $D(X)$, X compact and basically disconnected. The class σBA of σ -complete Boolean algebras consists exactly of the clopen algebras $\text{clop}X$ for the same X [16]. So, the various properties of EC 's considered here have direct translations to σBA . For example, corresponding to 4.6 are the σBA 's of the form $\mathcal{A} \in \bigcup_{\omega} \{\mathcal{B}(T) \mid T \text{ closed in } \mathbb{R}^\omega\}$, \mathcal{B} denoting the σ -field of Baire sets.

We leave the subject for now.

Acknowledgement

We thank Mojgan Mahmoudi for assistance in preparation of the paper.

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