Representation of $H$-closed monoreflections in archimedean $\ell$-groups with weak unit

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Abstract. The category of the title is called $\mathcal{W}$. This has all free objects $F(I)$ ($I$ a set). For an object class $\mathcal{A}$, $H\mathcal{A}$ consists of all homomorphic images of $\mathcal{A}$-objects. This note continues the study of the $H$-closed monoreflections $(\mathcal{R}, r)$ (meaning $HR = \mathcal{R}$), about which we show (inter alia): $A \in \mathcal{A}$ if and only if $A$ is a countably up-directed union from $H\{rF(\omega)\}$. The meaning of this is then analyzed for two important cases: the maximum essential monoreflection $r = c^3$, where $c^3F(\omega) = C(\mathbb{R}^\omega)$, and $C \in H\{c(\mathbb{R}^\omega)\}$ means $C = C(T)$, for $T$ a closed subspace of $\mathbb{R}^\omega$; the epicomplete, and maximum, monoreflection, $r = \beta$, where $\beta F(\omega) = B(\mathbb{R}^\omega)$, the Baire functions, and $E \in H\{B(\mathbb{R}^\omega)\}$ means $E$ is an epicompletion (not “the”) of such a $C(T)$.

1 Introduction

$\mathcal{W}$ is the category of archimedean $\ell$-groups $G$ with distinguished weak order unit $e_G$, and morphisms $G \xrightarrow{\varphi} H$ the $\ell$-group homomorphisms with $\varphi(e_G) =$
eH. We compress the discussion in §1 of [11], which see for more detail. “A ≤ B” means A is a W-subobject of B.

The forgetful functor \( \mathcal{W} \to \text{Sets} \) has the left adjoint \( F \). An \( F(I) \) is the free object on the set \( I \), and this is the \( \mathcal{W} \)-subobject of \( \mathbb{R}^I \) generated by the constant function 1, and all projections \( \pi_i : \mathbb{R}^I \to \mathbb{R} \) (\( i \mapsto \pi_i \) is the “insertion of generators” \( I \hookrightarrow F(I) \)).

A full subcategory \( \mathcal{R} \) of \( \mathcal{W} \) is monoreflective if \( \forall A \in \mathcal{W} \exists \text{monic } A \overset{r_A}{\to} rA, rA \in \mathcal{R} \), with the property: \( \forall A \overset{\varphi}{\to} R, R \in \mathcal{R}, \exists! rA \overset{\varphi r_A}{\to} R \) with \( \varphi r_A = \varphi \). We usually write \( A \leq rA \) for the \( rA \).

We abuse language and notation by saying as convenient \((\mathcal{R}, r)\) or \( \mathcal{R} \), or \( r \), is a monoreflection.

The class of monoreflections is ordered by: \( r \leq s \) means \( \forall A \exists \text{monic } f \) with \( sf = fr \).

“\( \omega \)” stands for the natural numbers, or any countable set, or the ordinal or cardinal.

**Theorem 1.1** ([11], 2.7). Suppose \((\mathcal{R}, r)\) is an \( H \)-closed monoreflection. Then \( \mathcal{R} = \text{inj} \{ F(\omega) \leq rF(\omega) \} \).

Theorem 1.1 is one of the main results of [11] and is the cornerstone of that paper. It devolves from categorical generalities, and many special features of \( \mathcal{W} \), some of which we describe below, and some later when needed.

Another main result of [11] is the characterization of the \( rF(\omega) \) in Theorem 1.1. Namely, 3.6 there says these are exactly the \( S \) with \( F(\omega) \leq S \leq B(\mathbb{R}^\omega) \) (\( B \) the Baire functions), with \( \sigma \) epic and \( S \circ S^\omega = S \) (that is, \( \forall s \) and countable \( \{s_n\} \) from \( S \), the function \( \mathbb{R}^\omega \overset{(s_n)}{\to} \mathbb{R}^\omega \overset{\sigma}{\to} \mathbb{R} \) lies in \( S \)). The cases for \( c^3 \) and \( \beta \) are mentioned in the Abstract, and will be deployed below.

Let \( \biguplus \) denote a countably up-directed union, in Sets or in \( \mathcal{W} \). For \( A \subseteq \text{Sets or } \mathcal{W} \), \( A \in \biguplus A \) means there is a family \( A' \) of \( A \)-subobjects of \( A \) with \( A = \biguplus A' \).

For \( I \in \text{Sets} \), let \( \mathcal{P}_0(I) = \{ J \subseteq I \mid |J| \leq \omega \} \). Then \( I = \biguplus \mathcal{P}_0(I) \). For \( A \in \mathcal{W} \), \( A = \biguplus \{ B \leq A \mid |B| \leq \omega \} \). From the form of the \( F(I) \), and the fact that any \( f \in C(\mathbb{R}^I) \) factors through a countable subproduct, we have \( F(I) = \biguplus \{ F(J) \mid J \in \mathcal{P}_0(I) \} \).
A crucial ingredient to what we have said so far, and necessary later, is the Yosida representation of $W$-objects:

$\mathbb{R}$ is the real numbers, and $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ under the obvious topology and order. For $X$ a topological space, $D(X) = \{ f \in C(X, \overline{\mathbb{R}}) \mid f^{-1}\mathbb{R}$ dense in $X\}$. This is a lattice containing $C(X)$, but has only partly defined $\oplus$. For $A \in W$, $A \leq D(X)$ means $A \cong A' \subseteq D(X)$, where $A'$ is closed under the partly defined data required to make $A' \in W$.

The Yosida representation of $A \in W$ (see [12]) says:

1. $A \leq D(YA)$ for a unique compact Hausdorff $YA$ for which $A$ separates the points.

2. For $A \stackrel{φ}{\rightarrow} B \in W$, there is a unique continuous $YA \stackrel{φ}{\leftarrow} YB$ for which $φ(a) = a \circ φ \forall a \in A$. If $φ$ is onto, then $φ$ is an embedding, $YA \leftrightarrow YB$.

The Yosida representation of $C(X)$, $X$ Tychonoff, is Čech-Stone extension $C(X) \ni f \mapsto βf \in D(βX)$.

2 Main Theorem

We expand on Theorem 1.1.

**Theorem 2.1.** Suppose $(\mathcal{R}, r)$ is an $H$-closed monoreflection in $W$. For $A \in W$, the following are equivalent:

1. $A \in \mathcal{R}$.
2. There is $I$ with a surjection $rF(I) \rightarrow A$.
3. $A \in \text{inj} \{ F(\omega) \leq rF(\omega) \}$.
4. Each countable $B \leq A$ ($i_B$ labels the inclusion) has the property
   
   $$\text{there is } rB \xrightarrow{i_B} A \text{ with } i_B rB = i_B. \quad (*)$$

5. $A \in \bigcup_{\omega} H\{rF(\omega)\}$.
6. $A \in \bigcup \mathcal{R}$.

**Proof.** (1)$\Leftrightarrow$(2) is quite general: For (1) $\implies$ (2), take $F(I) \rightarrow A$. We have $rF(I) \xrightarrow{φ} A$ with $φ rF(I) = φ$ because $A \in \mathcal{R}$, and $φ$ is a surjection.
(2) \implies (1) because \( \mathcal{R} = H\mathcal{R} \).
(1) \iff (3) is exactly Theorem 1.1.
(1) \implies (2) is obvious (in fact, for any \( B \leq A \)).
(4) \implies (5). We isolate two steps of the proof, just assuming \((\mathcal{R}, r)\) monoreflective (not assuming \((H\mathcal{R} = \mathcal{R})\). Proofs of these items are obvious.

Step (i). Suppose \( B \in \mathcal{W} \) and \( |B| \leq \omega \). Take any \( F(\omega) \xrightarrow{\varphi} B \). We then take \( \varphi \) as shown
\[
\begin{array}{c}
F(\omega) \leq rF(\omega) \\
\varphi \\
B \\
rB
\end{array}
\]
commuting, so \( B \leq \varphi(rF(\omega)) \leq rB \).

Step (ii). Suppose \( A = \bigcup \alpha B_\alpha \), where each \( B_\alpha \leq A \) has the property (*) in (4), with corresponding \( \tilde{r}_B \). Then \( A = \bigcup \alpha \tilde{r}_B \tilde{r}B_\alpha \).

Now suppose \( H\mathcal{R} = \mathcal{R} \). In Step (i), we then have \( \varphi(rF(\omega)) \in \mathcal{R} \), thus \( \overline{\varphi}_R(rF(\omega)) = rB \), because the embedding \( B \leq rB \) is “minimal to \( \mathcal{R} \)” (see [10]). This makes \( rB \in H\{rF(\omega)\} \).

Finally: Write \( A = \bigcup \alpha \{ B \mid B \leq A, |B| \leq \omega \} \). By (4), step (ii) applies and \( A = \bigcup \alpha \{ \tilde{r}_B(rB) \mid B \leq A, |B| \leq \omega \} \). Since each \( rB \in H\{rF(\omega)\} \), also each \( \tilde{r}_B(rB) \in H\{rF(\omega)\} \). Thus, (5).

(5) \implies (6) because \( H\mathcal{R} = \mathcal{R} \).

(6) \implies (3) This amounts to showing that \( \text{inj} \{ F(\omega) \leq rF(\omega) \} \) is closed under \( \bigcup \), since we already noted (3) \iff (1). So suppose \( A = \bigcup \alpha R_\alpha \), \( R_\alpha = \text{inj} \{ F(\omega) \leq rF(\omega) \} \), and take \( F(\omega) \xrightarrow{\varphi} A \). Since \( |F(\omega)| = \omega \), also \( |\varphi(F(\omega))| \leq \omega \), and \( \varphi(F(\omega)) \leq \text{some } R_\alpha \). So there is \( rF(\omega) \xrightarrow{\varphi} R_\alpha \in A \) extending \( \varphi \).

We now examine 2.1 for the important cases \( r = c^3 \) and \( r = \beta \).

3 \quad c^3 (Closed under countable composition)

"c^3" stands for “closed under countable composition”, originally studied in [13]. The definition goes as follows.

Each \( A \in \mathcal{W} \) has its Yosida representation \( A \leq D(\mathcal{Y}A) \). A sequence \( a_1, a_2, \ldots \) from \( A \) has \( \bigcap a^{-1}_i \mathcal{R} \) dense in \( \mathcal{Y}A \) (Baire Category Theorem) and
let \( \langle a_n \rangle = \bigcap_n a_n^{-1}\mathbb{R} \to \mathbb{R}^\omega \) be the function defined by \( \pi_j(\langle a_n \rangle(x)) = a_j(x) \) \( \forall j \). For \( f \in C(\mathbb{R}^\omega) \), we have the composition \( \bigcap_m a_n^{-1}(\mathbb{R}) \overset{\langle a_n \rangle}{\to} \mathbb{R}^\omega \overset{f}{\to} \mathbb{R} \). \( A \) is \( c^3 \) if each such \( f \circ \langle a_n \rangle \) extends over \( \mathcal{Y} \) to an element of \( A \).

\( c^3 \) will denote either the object class, or reflections \( A \leq c^3 A \). We assemble known facts.

**Theorem 3.1.** (Each item without specific reference can be located in [11] §1, with reference to original sources.)

(a) ([13]). \( A \) is \( c^3 \) if and only if \( A = \bigcup \{ C(\bigcap_n a_n^{-1}\mathbb{R}) \mid a_1, a_2, \cdots \in A \} \) if and only if there is a Tychonoff space \( X \) and a surjection \( C(X) \to A \).

(b) \( A \) is \( c^3 \) if and only if \( A \approx C(\mathcal{L}) \), \( \mathcal{L} \) a locale (aka, the \( f \)-ring of real-valued continuous functions on a frame \( \mathcal{L} \)).

(c) \( c^3 \) is monoreflective, with reflections \( A \leq c^3 A = \lim\{ C(\bigcap_n a_n^{-1}\mathbb{R}) \mid a_1, a_2, \cdots \in A \} \), and is an essential monoreflection (meaning that \( A \leq c^3 A \) is a essential monic).

The class \( c^3 \) is \( H \)-closed.

(d) \( c^3 \) is the largest essential monoreflection (with the smallest class of objects).

(e) \( \forall \) set \( I \), \( c^3 F(I) = C(\mathbb{R}^I) = \bigcup \{ C(\mathbb{R}^J) \mid J \in \mathcal{P}_0(I) \} \).

We consider the meaning of 2.1 (4) for \( r = c^3 \).

A Tychonoff space \( X \) is called Čech-complete if \( X \) is \( G_\delta \) in \( \beta X \) (see [7]). We abbreviate “Lindelöf and Čech-complete” to “LČ”.

**Theorem 3.2.** If \( X \) is LČ, then \( H\{C(X)\} = \{C(T) \mid T \text{ closed in } X \} \).

**Proof.** First note: For any Tychonoff space \( X \) and \( T \subseteq X \), the restriction \( C(X) \ni f \mapsto f|T \in C(T) \) defines a \( W \)-homomorphism \( C(X) \overset{\rho_T}{\to} C(T) \), and \( \rho_T \) is onto if and only if \( T \) is \( C \)-embedded in \( X \) (which entails the closure \( \overline{T} \) is \( C \)-embedded.) (See [8].)

Now suppose \( X \) is LČ. Then \( X \) is normal, so any closed \( T \) is \( C \)-embedded, thus \( C(T) \in H\{C(X)\} \).

For the converse, we shall use details of the Yosida representation; see §1. Any \( A \overset{\varphi}{\to} B \) has the quasi-dual embedding \( \mathcal{Y} A \leftarrow \mathcal{Y} B \) for which \( \varphi(a) = a|\mathcal{Y} B \) \( \forall a \in A \). This entails \( a^{-1}\mathbb{R} \cap \mathcal{Y} B \) dense in \( \mathcal{Y} B \), and thus \( \forall a_1, a_2, \cdots \in A \), \( \bigcap_n a_n^{-1}(\mathbb{R}) \) \( \cap \mathcal{Y} B = \bigcap_n (a_n^{-1}\mathbb{R} \cap \mathcal{Y} B) \) dense in \( \mathcal{Y} B \) (Baire Category Theorem).
Now, the Yosida representation of a $C(X)$ is extension over Čech-Stone compactification $\beta X$, as $C(X) \approx \{ \beta a \mid a \in C(X) \}$. And $X$ is LČ if and only if $\exists a_1, a_2, \cdots \in C(X)$ with $X = \bigcap_n (\beta a_n)^{-1} \mathbb{R}$.

Suppose $X$ is LČ, and $C(X) \xrightarrow{\varphi} B$ with Yosida dual embedding $\beta X \hookrightarrow \mathcal{Y}B$. Take $\{a_n\} \subseteq C(X)$ with $X = \bigcap_n (\beta a_n)^{-1} \mathbb{R}$ as above. Then $T = X \cap \mathcal{Y}B = \bigcap_n (\beta a_n)^{-1} \mathbb{R} \cap \mathcal{Y}B$ is dense in $\mathcal{Y}B$. (So we can view $B \leq C(T)$), and closed in the normal $X$ (thus $C$-embedded) so $B = C(T)$. \hfill \Box

Summing up, we interpret Theorem 2.1 for $c^3$ through Theorem 3.2 and some of Theorem 3.1.

**Corollary 3.3.** For $A \in \mathcal{W}$, the following are equivalent:

1. $A \in c^3$.
2. There is $I$ with a surjection $C(\mathbb{R}^I) \twoheadrightarrow A$.
3. $A \in \text{inj}\{ F(\omega) \leq C(\mathbb{R}^\omega) \}$.
4. For any countable $B \leq A$, also $c^3 B \leq A$.
5. $A \subset \bigcup \{ C(T) \mid T \text{ closed in } \mathbb{R}^\omega \}$.
6. $A \in \bigcup c^3$.

**Proof.** This is all quite immediate. We just note: (4) is just Theorem 2.1(5), using Theorem 3.2 for $X = \mathbb{R}^\omega$.

(4) is the statement that in Theorem 2.1(4) the $7_B$ are one-to-one. This follows solely from the essentiality of the reflection maps $B \leq c^3 B$. \hfill \Box

**Remark 3.4.**

(a) $A \subset \bigcup c^3 \iff A \subset \bigcup c^3$. An example is $A = \{ f \in C(\mathbb{R}^\omega) \mid \exists \text{ finite } F \subseteq \omega \text{ s.t. } f = \overline{f} \circ \pi_F \}$.

(b) Corollary 3.3 (3) and (5) are to be compared with Theorem 3.1(a). The $X$ in Theorem 3.1(a) is $\mathcal{Y}A \times \mathbb{N}$.

(c) We note [7], p. 74: $T$ is ($\approx$) a closed subspace of $\mathbb{R}^\omega$ if and only if $T$ is completely metrizable and separable.

(d) In Corollary 3.3 (5), the $A = \bigcup C(T)$’s is a countably directed direct limit, $A = \operatorname{lim}_{\leftarrow \omega} C(T)$’s. The Yosida functor converts this to an inverse limit $\mathcal{Y}A = \operatorname{lim}_{\leftarrow \omega} \beta T$’s. Using $A = C(X)$ with $X$ real compact, and a little fiddling yields $X = \operatorname{lim}_{\leftarrow \omega} T$’s, and if $X$ is compact, so are the $T$’s. This is more or less a result of Pasynkov [15]. See also [7], p. 220.

(e) An essential reflection $(\mathcal{R}, r)$ has $r \leq c^3$ (Theorem 3.1 (d)), and if $\mathcal{R} = H\mathcal{R}$, Corollary 3.3 holds mutatis mutandis. For $r F(\omega) = S$ (see
the second paragraph after Theorem 1.1), we have $F(\omega) \leq S \leq C(\mathbb{R}^\omega)$, and “$\sigma$ epic” is automatic. Examples of this are: $\mathcal{R} = \text{“rings”}$ ($\mathcal{W}$-objects $A$ with a compatible $f$-ring multiplication with identity the $\mathcal{W}$-unit $e_A$), vector lattices, algebras, . . . . For example: for rings, $rF(\omega)$ is the sub-$f$-ring of $C(\mathbb{R}^\omega)$ generated by $F(\omega)$. In Corollary 3.3 (4), each $C(T)$ is to be replaced by the set of restrictions $rF(\omega)|T$. An additional feature of any essential $r$ is that $rF(\omega)|T = r(F(\omega)|T)$.

(f) The present paper began with an analysis of a version of Corollary 3.3 and some related matters, in the view of a $c^2$-object as the $f$-ring of real-valued continuous functions on a frame. As such, it was reported in [6]: where $c^2$ was taken as condition 3.3(3), thus avoiding a reference to the Yosida representation and the reflection is then given an explicit frame-theoretic form. See [4] for details.

4 $\beta$ (Epicomplete)

$E$ is called epicomplete if $E \xrightarrow{\varphi} \bullet$ monic and epic implies $\varphi$ an isomorphism. The class of epicomplete objects is denoted $EC$.

Recall that, for a Tychonoff space $X$, $B(X)$ denotes the $\mathcal{W}$-object of real-valued Baire functions on $X$.

We summarize known features of $EC$, prior to the interpretation of Theorem 2.1 for $\mathcal{R} = EC$.

**Theorem 4.1.** (Each item without specific reference can be located in [11], with reference to original sources.)

(a) $E \in EC$ if and only if $E$ is $\sigma$-complete both conditionally, and laterally if and only if $E \approx D(X)$ with $X$ basically disconnected (the $X$ is $\mathcal{Y}E$). Thus, any $B(X) \in EC$.

(b) ([3]). $E \approx C(\mathcal{P})$ with $\mathcal{P}$ a $P$-locale. (Such a $\mathcal{P}$ is the localic intersection of $\{S \mid S$ is dense cozero in $\mathcal{Y}E\}$.)

(c) $EC$ is monoreflective, thus the maximum monoreflection. The reflection of $A$ is $\beta A = B(\mathcal{Y}A)/N$, for a certain $\sigma$-ideal $N$.

$EC$ is $H$-closed, thus $EC = H\{B(K) \mid K$ compact$\}$.

(d) If $X$ is Lindelöf and Čech-complete, then $\beta C(X) = B(X)$.

(e) For every set $I$, $\beta F(I) = B(\mathbb{R}^I) = \bigcup_{\omega} B(\mathbb{R}^J) \mid J \in \mathcal{P}_0(I)$.
We now interpret Theorem 2.1. Most of this is the routine writing-down of items in Theorem 2.1 using information in Theorem 4.1. An exception is Theorem 2.1 (5), which says $A \in H\{B(\mathbb{R}^\omega)\}$. “An” epicompletion of $A \in \mathcal{W}$ is an epic $A \leq E$, with $E \in EC$. These are exactly the quotients over $A$ of $\beta A$.

**Theorem 4.2.** Suppose $X$ is $L\check{C}$ (as is $\mathbb{R}^\omega$).

(a) $E \in H\{B(X)\}$ if and only if there is $F$ closed in $X$ such that $E$ is an epicompletion of $C(F)$.

(b) (Note that an $F$ in (a) is again $L\check{C}$.) $C(X)$ has a unique epicompletion if and only if $X$ is discrete and countable (and thus $X \approx \mathbb{N}$, $C(X) \approx C(\mathbb{N})$, is already $EC$).

(c) If $X$ is not countable discrete, there are many epicompletions of $C(X)$.

**Proof.** (a) Suppose $E \in H\{B(X)\}$, as $B(X) \overset{\varphi}{\twoheadrightarrow} E$. We have

\[
\begin{array}{c}
C(X) \overset{\beta C}{\leq} \beta C(X) = B(X) \text{ (by Theorem 4.1(d))} \\
\varphi_0 \downarrow \quad \varphi \\
\varphi(C(X)) \overset{e}{\leq} E
\end{array}
\]

where $\varphi_0$ is the restriction of $\varphi$, $e$ labels the inclusion, and $\varphi \beta C = e \varphi_0$ (obviously), so $e$ is epic (as a second factor of the epic $\varphi \beta C$).

By Theorem 3.2, $\varphi(C(X))$ is the desired $C(F)$.

Suppose $F$ is closed in $X$ and $C(F) \overset{e}{\leq} E$ is an epicompletion. We then have

\[
\begin{array}{c}
C(X) \overset{\beta C}{\leq} \beta C(X) \\
\rho \downarrow \quad \rho \\
C(F) \overset{e}{\leq} E
\end{array}
\]

where $\rho$ is the restriction map described at the beginning of the proof of Theorem 3.2, and then $\exists \rho$ with $\rho \beta C = e \rho$ by the universal mapping property of $\beta$.

We have $C(F) \overset{i}{\leq} \bar{\rho}(\beta C(X)) \overset{j}{\leq} E$ ($i, j$ are labels) with $ji = e$. But $\bar{\rho}(\beta C(X)) \in EC$ (by Theorem 4.1(c)), and $e$ is epic, thus also $j$. So $j$ is equality.
(b) If \( C(X) \approx C(\mathbb{N}) \), already \( C(X) \in EC \), so is its only epicompletion. If \( C(X) \) has a unique epicompletion, it must be \( C(X) \leq B(X) \) (Theorem 4.1 (d)), and this must be an essential embedding (because any \( A \in W \) has a (unique) essential epicompletion ( [2], §9)). If \( X \) has a non-void nowhere dense zero-set \( Z \), then the characteristic function \( \chi(Z) \in B(X) \), and there is no \( 0 < a \in C(X) \) with \( a \leq \chi(Z) \): \( C(X) \leq B(X) \) is not essential. Thus there is no such \( Z \), so \( X \) is what is called an almost \( P \)-space. But the only almost \( P \)-space which is \( L\overline{C} \) is \( \approx \mathbb{N} \).

(c) See [1] and [2] for several constructions. We omit details. 

Referring to Theorem 4.2, let \( \mathcal{ECS}(\mathbb{R}^\omega) \) stand for the family of epicompletions of objects of the form \( C(T) \), for \( T \) closed in \( \mathbb{R}^\omega \).

Summing up, we write down Theorem 2.1 for \( W \overset{\beta}{\rightarrow} EC \) using Theorem 4.2 and some of Theorem 4.1.

**Corollary 4.3.** For \( A \in W \), the following are equivalent:

1. \( A \in EC \).
2. There is \( I \) with a surjection \( B(\mathbb{R}^I) \twoheadrightarrow A \).
3. \( A \in \text{inj} \{ F(\omega) \leq B(\mathbb{R}^\omega) \} \).
4. Each countable \( B \leq A \), has the property
   \[
   \text{there is } \beta B \overset{i_B}{\rightarrow} A \text{ with } \beta B i_B = i_B. \tag{*}
   \]
5. \( A \in \bigcup \mathcal{ECS}(\mathbb{R}^\omega) \).
6. \( A \in \bigcup EC \).

The comparison of Corollary 4.3 (4) and (5) with Corollary 3.3 (4) and (5), shows a huge difference between \( c^3 \) (or any essential reflection) with \( \beta \) and identifies some special classes of \( EC \) objects which might deserve further study. (It is quite rare that any \( A \leq \beta A \) is essential; see [2], §9.)

We consider the analogue of Corollary 3.3 (4) for \( \beta \). Recall that for \( B \leq A \), \( \beta B \leq A \) means that the \( \overline{i_B} \) in 2.8 (4) is one-to-one.

**Theorem 4.4.** Suppose \( A \in W \). For every countable \( B \leq A \),

\[
\beta B \leq A \text{ if and only if } A \approx \mathbb{R}^n \text{ for some } n \in \mathbb{N}. \tag{*}
\]
Proof. Notice that either the condition implies $A \in EC$: for $A \in EC$ (or just a vector lattice), $A \approx \mathbb{R}^n$ ($n \in \mathbb{N}$) means $|\mathcal{Y}A| = n$ (and $\mathbb{R}^n = C(\{0, 1, \ldots, n-1\})$).

$(\Leftarrow)$ We omit the easy proof.

$(\Rightarrow)$ We show that $\mathcal{Y}A$ infinite $\implies A$ fails $(\ast)$.

(i) $A = C(\mathbb{N})$ fails $(\ast)$.

(ii) If $A \in EC$ and $\mathcal{Y}A$ is infinite, then there is an embedding $C(\mathbb{N}) \leq A$.

(iii) If $A \in EC$ and $\mathcal{Y}A$ is infinite, then $A$ fails $(\ast)$.

For (i): Let $B \leq C(\mathbb{N})$ be generated by rational multiples of the characteristic functions $\chi_p$ of the $p \in \mathbb{N}$. A little thought reveals that the uniform completion $\overline{uB} = c^3B = C(\alpha \mathbb{N})$, where $\alpha \mathbb{N} = \mathbb{N} \cup \{\alpha\}$ the one-point compactification of $\mathbb{N}$. Then $\beta C(\alpha \mathbb{N}) = B(\alpha \mathbb{N})$ (Theorem 4.1 (d)). Then, the $\overline{i}_B$ is not one-to-one: $\overline{i}_B(\psi_{\alpha}) = 0$.

For (ii): As with any infinite Hausdorff space, there is countable $L = \{x_n\} \subseteq \mathcal{Y}A$ on pairwise disjoint open sets $\{U_n\}$ in $\mathcal{Y}A$ with $U_n \cap L = \{x_n\}$ $\forall n$. We have $L \approx \mathbb{N}$. Since $\mathcal{Y}A$ is basically disconnected, thus zero-dimensional ( [8]). The $U_n$ may be chosen clopen, and $\overline{U} \approx \beta \overline{U}$ (Čech-Stone). Choose any $p_0 \in U$, and retract $\mathcal{Y}A \overset{\rho}{\rightarrow} U$ as $\rho(x) = [x, \text{if } x \in U; p_0 \text{ if } x \notin U]$. Since $\overline{U}$ is clopen, $\rho$ is continuous.

Let $f \in C(L)$. Extend to $\overline{f} \in C(U)$ by $f(U_n) = \{f(x_n)\}$. Then extend $\overline{f}$ to $\overline{f} \in D(\beta \overline{U})$ (since $\overline{U} = \beta U$). Now $\overline{f} \circ \rho \in D(\mathcal{Y}A)$ ($(\overline{f} \circ \rho)^{-1} \mathbb{R} = \overline{f}^{-1}(\mathbb{R}) \subseteq \overline{U} - u$, which is nowhere dense). Define $C(L) \overset{\hat{\rho}}{\leq} D(\mathcal{Y}A) \in A$ as $\hat{\rho}(f) = \overline{f} \circ \rho$. Such compositions preserve ℓ-group operations (and multiplication) and constants, so $\hat{\rho}(1) = 1$, and $\hat{\rho} \in \mathcal{W}$.

For (iii): In (ii) we have $C(L) \overset{\hat{i}}{\leq} A$, which we re-name $C(\mathbb{N}) \overset{k}{\leq} A$. In (i), we have countable $B \overset{i_B}{\leq} C(\mathbb{N})$ with $\overline{i}_B$ not 1-1. We have

\[
\begin{array}{ccc}
B & \overset{i_B}{\leq} & C(\mathbb{N}) \\
\beta B & \overset{\overline{i}_B}{\leq} & A \\
\end{array}
\]
with the inclusion $B^j \leq A$ being $j_B = ki_B$ and with $\tilde{i}_B \beta_B = i_B, \tilde{j}_B \beta_B = j_B = ki_B$. Thus $\tilde{j}_B \beta_B = k\tilde{i}_B \beta_B$, so $\tilde{j}_B = k\tilde{i}_B$ since $\beta_B$ is epic. Since $\tilde{i}_B$ is not one-to-one, neither is $\tilde{j}_B$. 

Let $BS(\mathbb{R}^\omega) \equiv \{B(T) \mid T \text{ dense in } \mathbb{R}^\omega\}$. The analogue of Corollary 3.3(5) for $\beta$ is the condition

$$A \in \check{\bigcup} \{B(T) \mid T \text{ closed in } \mathbb{R}^\omega\}.$$  (**)

All we have to say is: sometimes this happens, sometimes not.

**Remark 4.5.** (a) There are $A$ satisfying (**): Obviously, any $B(T)$; less trivially, ([11]) for uncountable $I$, $B(\mathbb{R}^I) = \check{\bigcup} \{B(\mathbb{R}^J) \mid J \in \mathcal{P}_0(I)\}$.

(b) There are many $A$ failing (**). The countable chain condition, ccc, of a space or $\mathcal{W}$-object is relevant here. $X$ (resp., $A$) has ccc if there is no uncountable pairwise disjoint family of non-void open sets in $X$ (respectively, non-zero positive elements in $A$). $A$ has ccc if and only if $\check{\mathcal{Y}}A$ does (because $\text{coz } A$ is a base in $\check{\mathcal{Y}}A$).

If $A$ has ccc and satisfies (**), then in the Yosida representation $A \approx D(\check{\mathcal{Y}}A)$, each $a \in A$ is locally constant on a dense open subset of $\check{\mathcal{Y}}A$. (If $A = \check{\bigcup} B(T_\alpha)$, then each $B(T_\alpha)$ has ccc, and it follows that $T_\alpha$ is a copy $N_\alpha$ of $N$. For each $\alpha$, $C(N_\alpha) = \beta N_\alpha \check{\hookrightarrow} \check{\mathcal{Y}}A$. If $a \in C(N_\alpha)$, then $a" =^" a \circ \tau$ is locally constant on $\tau^{-1}(N_\alpha)$.)

Consider the absolute (projective cover) $[0,1] \xleftarrow{\pi} a[0,1]$. Using irreducibility of $\pi$: Since $[0,1]$ has ccc, so do $a[0,1]$, and also $A = D(a[0,1])$. Here $C([0,1]) \leq A$, as $f \mapsto f \circ \pi$. No continuous nonconstant $f$ has $f \circ \pi$ locally constant on a dense subset of $a[0,1]$. Thus $A$ fails (**).

(c) The class $EC$ consists exactly of the $D(X)$, $X$ compact and basically disconnected. The class $\sigma BA$ of $\sigma$-complete Boolean algebras consists exactly of the clopen algebras $\text{clop } X$ for the same $X$ [16]. So, the various properties of $EC$’s considered here have direct translations to $\sigma BA$. For example, corresponding to 4.6 are the $\sigma BA$’s of the form $A \in \check{\bigcup} \{B(T) \mid T \text{ closed in } \mathbb{R}^\omega\}, B$ denoting the $\sigma$-field of Baire sets.

We leave the subject for now.
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