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Representation of H-closed monoreflections in archimedean ℓ-groups with weak unit

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Abstract. The category of the title is called $\mathcal{W}$. This has all free objects $F(I)$ (I a set). For an object class $\mathcal{A}$, $H\mathcal{A}$ consists of all homomorphic images of $\mathcal{A}$-objects. This note continues the study of the H-closed monoreflections $(\mathcal{R}, r)$ (meaning $H\mathcal{R} = \mathcal{R}$), about which we show (*inter alia*): $A \in \mathcal{A}$ if and only if A is a countably up-directed union from $H\{rF(\omega)\}$. The meaning of this is then analyzed for two important cases: the maximum essential monoreflection $r = c^3$, where $c^3F(\omega) = C(\mathbb{R}^\omega)$, and $C \in H\{c(\mathbb{R}^\omega)\}$ means $C = C(T)$, for T a closed subspace of $\mathbb{R}^\omega$; the epicomplete, and maximum, monoreflection, $r = \beta$, where $\beta F(\omega) = B(\mathbb{R}^\omega)$, the Baire functions, and $E \in H\{B(\mathbb{R}^\omega)\}$ means E is *an* epicompletion (not "the") of such a $C(T)$.

1 Introduction

$\mathcal{W}$ is the category of archimedean ℓ-groups G with distinguished weak order unit e_G, and morphisms $G \xrightarrow{\varphi} H$ the ℓ-group homomorphisms with $\varphi(e_G) =$

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e_H. We compress the discussion in §1 of [11], which see for more detail. "$A \leq B$" means A is a $\mathcal{W}$-subobject of B.

The forgetful functor $\mathcal{W} \to$ Sets has the left adjoint F. An $F(I)$ is the free object on the set I, and this is the $\mathcal{W}$-subobject of $\mathbb{R}^{\mathbb{R}^I}$ generated by the constant function 1, and all projections $\pi_i : \mathbb{R}^I \to \mathbb{R}$ ($i \mapsto \pi_i$ is the "insertion of generators" $I \hookrightarrow F(I)$).

A full subcategory $\mathcal{R}$ of $\mathcal{W}$ is monoreflective if $\forall A \in \mathcal{W}\ \exists$ monic $A \xrightarrow{r_A} rA$, $rA \in \mathcal{R}$, with the property: $\forall A \xrightarrow{\varphi} R$, $R \in \mathcal{R}$, $\exists!\, rA \xrightarrow{\overline{\varphi}} R$ with $\overline{\varphi} r_A = \varphi$. We usually write $A \leq rA$ for the r_A. We abuse language and notation by saying as convenient $(\mathcal{R}, r)$ or $\mathcal{R}$, or r, is a monoreflection.

The class of monoreflections is ordered by: $r \leq s$ means $\forall A\ \exists$ monic f with $s_A = f r_A$.

Let $M \xrightarrow{m} M' \in \mathcal{W}$. Then, $A \in \operatorname{inj}\{m\}$ means: $\forall M \xrightarrow{\varphi} A\ \exists M' \xrightarrow{\varphi'} A$ with $\varphi' m = \varphi$.

"ω" stands for the natural numbers, or any countable set, or the ordinal or cardinal.

Theorem 1.1 ([11], 2.7). *Suppose $(\mathcal{R}, r)$ is an H-closed monoreflection. Then $\mathcal{R} = \operatorname{inj}\{F(\omega) \leq rF(\omega)\}$.*

Theorem 1.1 is one of the main results of [11] and is the cornerstone of that paper. It devolves from categorical generalities, and many special features of $\mathcal{W}$, some of which we describe below, and some later when needed.

Another main result of [11] is the characterization of the $rF(\omega)$ in Theorem 1.1. Namely, 3.6 there says these are exactly the S with $F(\omega) \overset{\sigma}{\leq} S \leq B(\mathbb{R}^\omega)$ (B the Baire functions), with σ epic and $S \circ S^\omega = S$ (that is, $\forall s$ and countable $\{s_n\}$ from S, the function $\mathbb{R}^\omega \xrightarrow{\langle s_n \rangle} \mathbb{R}^\omega \xrightarrow{s} \mathbb{R}$ lies in S). The cases for c^3 and β are mentioned in the Abstract, and will be deployed below.

Let $\overset{\omega}{\Uparrow\!\bigcup}$ denote a countably up-directed union, in Sets or in $\mathcal{W}$. For $\mathcal{A} \subseteq$ Sets or $\mathcal{W}$, $A \in \overset{\omega}{\Uparrow\!\bigcup} \mathcal{A}$ means there is a family $\mathcal{A}'$ of $\mathcal{A}$-subobjects of A with $A = \overset{\omega}{\Uparrow\!\bigcup} \mathcal{A}'$.

For $I \in$ Sets, let $\mathcal{P}_0(I) = \{J \subseteq I \mid |J| \leq \omega\}$. Then $I = \overset{\omega}{\Uparrow\!\bigcup} \mathcal{P}_0(I)$. For $A \in \mathcal{W}$, $A = \overset{\omega}{\Uparrow\!\bigcup}\{B \leq A \mid |B| \leq \omega\}$. From the form of the $F(I)$, and the fact that any $f \in C(\mathbb{R}^I)$ factors through a countable subproduct, we have $F(I) = \overset{\omega}{\Uparrow\!\bigcup}\{F(J) \mid J \in \mathcal{P}_0(I)\}$.

A crucial ingredient to what we have said so far, and necessary later, is the Yosida representation of $\mathcal{W}$-objects:

$\mathbb{R}$ is the real numbers, and $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ under the obvious topology and order. For X a topological space, $D(X) = \{f \in C(X, \overline{\mathbb{R}}) \mid f^{-1}\mathbb{R}$ dense in $X\}$. This is a lattice containing $C(X)$, but has only partly defined $+$. For $A \in \mathcal{W}$, $A \leq D(X)$ means $A \overset{\mathcal{W}}{\approx} A' \subseteq D(X)$, where A' is closed under the partly defined data required to make $A' \in \mathcal{W}$.

The Yosida representation of $A \in \mathcal{W}$ (see [12]) says:

(1) $A \leq D(\mathcal{Y}A)$ for a unique compact Hausdorff $\mathcal{Y}A$ for which A separates the points.

(2) For $A \xrightarrow{\varphi} B \in \mathcal{W}$, there is a unique continuous $\mathcal{Y}A \xleftarrow{\mathcal{Y}\varphi} \mathcal{Y}B$ for which $\varphi(a) = a \circ \mathcal{Y}\varphi \ \forall a \in A$. If φ is onto, then $\mathcal{Y}\varphi$ is an embedding, $\mathcal{Y}A \hookleftarrow \mathcal{Y}B$.

The Yosida representation of $C(X)$, X Tychonoff, is Čech-Stone extension $C(X) \ni f \mapsto \beta f \in D(\beta X)$.

2 Main Theorem

We expand on Theorem 1.1.

Theorem 2.1. *Suppose $(\mathcal{R}, r)$ is an H-closed monoreflection in $\mathcal{W}$. For $A \in \mathcal{W}$, the following are equivalent:*

(1) $A \in \mathcal{R}$.

(2) *There is I with a surjection $rF(I) \twoheadrightarrow A$.*

(3) $A \in \operatorname{inj}\{F(\omega) \leq rF(\omega)\}$.

(4) *Each countable $B \overset{i_B}{\leq} A$ (i_B labels the inclusion) has the property*

$$\text{there is } rB \overset{\bar{i}_B}{\dashrightarrow} A \text{ with } \bar{i}_B r_B = i_B. \qquad (*)$$

(5) $A \in \overset{\omega}{\Cup} H\{rF(\omega)\}$.

(6) $A \in \overset{\omega}{\Cup} \mathcal{R}$.

Proof. (1)$\Leftrightarrow$(2) is quite general: For (1) $\implies$ (2), take $F(I) \overset{\varphi}{\twoheadrightarrow} A$. We have $rF(I) \xrightarrow{\overline{\varphi}} A$ with $\overline{\varphi} r_{F(I)} = \varphi$ because $A \in \mathcal{R}$, and $\overline{\varphi}$ is a surjection.

(2) $\Longrightarrow$ (1) because $\mathcal{R} = H\mathcal{R}$.

(1)$\Leftrightarrow$(3) is exactly Theorem 1.1.

(1) $\Longrightarrow$ (2) is obvious (in fact, for any $B \leq A$).

(4) $\Longrightarrow$ (5). We isolate two steps of the proof, just assuming $(\mathcal{R}, r)$ monoreflective (not assuming $(H\mathcal{R} = \mathcal{R})$. Proofs of these items are obvious.

Step (i). Suppose $B \in \mathcal{W}$ and $|B| \leq \omega$. Take any $F(\omega) \xrightarrow{\varphi} B$. We then take $\overline{\varphi}$ as shown

$$\begin{tikzcd} F(\omega) \arrow[d, "\varphi"'] & \leq rF(\omega) \arrow[d, dashed, "\overline{\varphi}"] \\ B & \leq rB \end{tikzcd}$$

commuting, so $B \leq \overline{\varphi}(rF(\omega)) \leq rB$.

Step (ii). Suppose $A = \overset{\omega}{\Uparrow\!\bigcup}_\alpha B_\alpha$, where each $B_\alpha \leq A$ has the property $(*)$ in (4), with corresponding $\bar{i}_{B_\alpha}$. Then $A = \overset{\omega}{\Uparrow\!\bigcup}_\alpha \bar{i}_{B_\alpha}(rB_\alpha)$.

Now suppose $H\mathcal{R} = \mathcal{R}$. In Step (i), we then have $\overline{\varphi}_R(rF(\omega)) \in \mathcal{R}$, thus $\overline{\varphi}_R(rF(\omega)) = rB$, because the embedding $B \leq rB$ is "minimal to $\mathcal{R}$" (see [10]). This makes $rB \in H\{rF(\omega)\}$.

Finally: Write $A = \overset{\omega}{\Uparrow\!\bigcup}\{B \mid B \leq A, |B| \leq \omega\}$. By (4), step (ii) applies and $A = \overset{\omega}{\Uparrow\!\bigcup}\{\bar{i}_B(rB) \mid B \leq A, |B| \leq \omega\}$. Since each $rB \in H\{rF(\omega)\}$, also each $\bar{i}_B(rB) \in H\{rF(\omega)\}$. Thus, (5).

(5) $\Longrightarrow$ (6) because $H\mathcal{R} = \mathcal{R}$.

(6) $\Longrightarrow$ (3) This amounts to showing that $\operatorname{inj}\{F(\omega) \leq rF(\omega)\}$ is closed under $\overset{\omega}{\Uparrow\!\bigcup}$, since we already noted (3) $\Longleftrightarrow$ (1). So suppose $A = \overset{\omega}{\Uparrow\!\bigcup} R_\alpha$, $R_\alpha = \operatorname{inj}\{F(\omega) \leq rF(\omega)\}$, and take $F(\omega) \xrightarrow{\varphi} A$. Since $|F(\omega)| = \omega$, also $|\varphi(F(\omega))| \leq \omega$, and $\varphi(F(\omega)) \leq$ some R_α. So there is $rF(\omega) \xrightarrow{\overline{\varphi}} R_\alpha \in A$ extending φ. $\square$

We now examine 2.1 for the important cases $r = c^3$ and $r = \beta$.

3 c^3 (Closed under countable composition)

"c^3" stands for "closed under countable composition", originally studied in [13]. The definition goes as follows.

Each $A \in \mathcal{W}$ has its Yosida representation $A \leq D(\mathcal{Y}A)$. A sequence $a_1, a_2, \ldots$ from A has $\bigcap a^{-1}\mathbb{R}$ dense in $\mathcal{Y}A$ (Baire Category Theorem) and let $\langle a_n \rangle = \bigcap_n a_n^{-1}\mathbb{R} \to \mathbb{R}^\omega$ be the function defined by $\pi_j(\langle a_n \rangle(x)) = a_j(x)$ $\forall j$. For $f \in C(\mathbb{R}^\omega)$, we have the composition $\bigcap_m a_n^{-1}(\mathbb{R}) \xrightarrow{\langle a_n \rangle} \mathbb{R}^\omega \xrightarrow{f} \mathbb{R}$. A is c^3 if each such $f \circ \langle a_n \rangle$ extends over $\mathcal{Y}A$ to an element of A.

c^3 will denote either the object class, or reflections $A \leq c^3 A$. We assemble known facts.

Theorem 3.1. (*Each item without specific reference can be located in* [11] §1, *with reference to original sources.*)

(a) ([13]). *A is c^3 if and only if $A = \overset{\omega}{\Uparrow}\{C(\bigcap_n a_n^{-1}\mathbb{R}) \mid a_1, a_2, \dots \in A\}$ if and only if there is a Tychonoff space X and a surjection $C(X) \twoheadrightarrow A$.*

(b) *A is c^3 if and only if $A \approx C(\mathcal{L})$, $\mathcal{L}$ a locale* (*aka, the f-ring of real-valued continuous functions on a frame $\mathcal{L}$*).

(c) *c^3 is monoreflective, with reflections $A \leq c^3 A = \varinjlim\{C(\bigcap_n a_n^{-1}\mathbb{R}) \mid a_1, a_2, \dots \in A\}$, and is an essential monoreflection* (*meaning that $A \leq c^3 A$ is a essential monic*).

The class c^3 is H-closed.

(d) *c^3 is the largest essential monoreflection* (*with the smallest class of objects*).

(e) $\forall$ *set I, $c^3 F(I) = C(\mathbb{R}^I) = \overset{\omega}{\Uparrow}\{C(\mathbb{R}^J) \mid J \in \mathcal{P}_0(I)\}$.*

We consider the meaning of 2.1 (4) for $r = c^3$.

A Tychonoff space X is called Čech-complete if X is G_δ in βX (see [7]).

We abbreviate "Lindelöf and Čech-complete" to "LČ".

Theorem 3.2. *If X is LČ, then $H\{C(X)\} = \{C(T) \mid T \text{ closed in } X\}$.*

Proof. First note: For any Tychonoff space X and $T \subseteq X$, the restriction $C(X) \ni f \mapsto f|T \in C(T)$ defines a $\mathcal{W}$-homomorphism $C(X) \xrightarrow{\rho_T} C(T)$, and ρ_T is onto if and only if T is C-embedded in X (which entails the closure $\overline{T}$ is C-embedded.) (See [8].)

Now suppose X is LČ. Then X is normal, so any closed T is C-embedded, thus $C(T) \in H\{C(X)\}$.

For the converse, we shall use details of the Yosida representation; see §1. Any $A \overset{\varphi}{\twoheadrightarrow} B$ has the quasi-dual embedding $\mathcal{Y}A \hookleftarrow \mathcal{Y}B$ for which $\varphi(a) = a|\mathcal{Y}B$ $\forall a \in A$. This entails $a^{-1}\mathbb{R} \cap \mathcal{Y}B$ dense in $\mathcal{Y}B$, and thus $\forall a_1, a_2, \dots \in A$, $\left(\bigcap_n a_n^{-1}(\mathbb{R})\right) \cap \mathcal{Y}B = \bigcap_n (a_n^{-1}\mathbb{R} \cap \mathcal{Y}B)$ dense in $\mathcal{Y}B$ (Baire Category Theorem).

Now, the Yosida representation of a $C(X)$ is extension over Čech-Stone compactification βX, as $C(X) \approx \{\beta a \mid a \in C(X)\}$. And X is LČ if and only if $\exists a_1, a_2, \cdots \in C(X)$ with $X = \bigcap_n (\beta a_n)^{-1}\mathbb{R}$.

Suppose X is LČ, and $C(X) \xrightarrow{\varphi} B$ with Yosida dual embedding $\beta X \hookleftarrow \mathcal{Y}B$. Take $\{a_n\} \subseteq C(X)$ with $X = \bigcap_n (\beta a_n)^{-1}\mathbb{R}$ as above. Then $T = X \cap \mathcal{Y}B = \bigcap_n (\beta a_n)^{-1}\mathbb{R} \cap \mathcal{Y}B$ is dense in $\mathcal{Y}B$. (So we can view $B \leq C(T)$), and closed in the normal X (thus C-embedded) so $B = C(T)$. □

Summing up, we interpret Theorem 2.1 for c^3 through Theorem 3.2 and some of Theorem 3.1.

Corollary 3.3. *For $A \in \mathcal{W}$, the following are equivalent:*

(1) $A \in c^3$.

(2) *There is I with a surjection $C(\mathbb{R}^I) \twoheadrightarrow A$.*

(3) $A \in \operatorname{inj}\{F(\omega) \leq C(\mathbb{R}^\omega)\}$.

(4) *For any countable $B \leq A$, also $c^3 B \leq A$.*

(5) $A \subseteq \overset{\omega}{\Uparrow\!\bigcup}\{C(T) \mid T$ *closed in* $\mathbb{R}^\omega\}$.

(6) $A \in \overset{\omega}{\Uparrow\!\bigcup} c^3$.

Proof. This is all quite immediate. We just note: (4) is just Theorem 2.1(5), using Theorem 3.2 for $X = \mathbb{R}^\omega$.

(4) is the statement that in Theorem 2.1(4) the $\bar{i}_B$ are one-to-one. This follows solely from the essentiality of the reflection maps $B \leq c^3 B$. □

Remark 3.4. (a) $A \in \Uparrow\!\bigcup\, c^3 \not\Rightarrow A \in \overset{\omega}{\Uparrow\!\bigcup} c^3$. An example is $A = \{f \in C(\mathbb{R}^\omega) \mid \exists$ finite $F \subseteq \omega$ s.t. $f = \overline{f} \circ \pi_F\}$.

(b) Corollary 3.3 (3) and (5) are to be compared with Theorem 3.1(a). The X in Theorem 3.1(a) is $\mathcal{Y}A \times \mathbb{N}$.

(c) We note [7], p. 74: T is ($\approx$) a closed subspace of $\mathbb{R}^\omega$ if and only if T is completely metrizable and separable.

(d) In Corollary 3.3 (5), the $A = \overset{\omega}{\Uparrow\!\bigcup} C(T)$'s is a countably directed direct limit, $A = \varinjlim_\omega C(T)$'s. The Yosida functor converts this to an inverse limit $\mathcal{Y}A = \varprojlim_\omega \beta T$'s. Using $A = C(X)$ with X real compact, and a little fiddling yields $X = \varprojlim_\omega T$'s, and if X is compact, so are the T's. This is more or less a result of Pasynkov [15]. See also [7], p. 220.

(e) An <u>essential</u> reflection $(\mathcal{R}, r)$ has $r \leq c^3$ (Theorem 3.1 (d)), and if $\mathcal{R} = H\mathcal{R}$, Corollary 3.3 holds *mutatis mutandis*. For $rF(\omega) = S$ (see the second paragraph after Theorem 1.1), we have $F(\omega) \overset{\sigma}{\leq} S \leq C(\mathbb{R}^\omega)$, and "$\sigma$ epic" is automatic. Examples of this are: $\mathcal{R}$ = "rings" ($\mathcal{W}$-objects A with a compatible f-ring multiplication with identity the $\mathcal{W}$-unit e_A), vector lattices, algebras, ... For example: for rings, $rF(\omega)$ is the sub-f-ring of $C(\mathbb{R}^\omega)$ generated by $F(\omega)$. In Corollary 3.3 (4), each $C(T)$ is to be replaced by the set of restrictions $rF(\omega)|T$. An additional feature of any essential r is that $rF(\omega)|T = r(F(\omega)|T)$.

(f) The present paper began with an analysis of a version of Corollary 3.3 and some related matters, in the view of a c^3-object as the f-ring of real-valued continuous functions on a frame. As such, it was reported in [6]: where c^3 was taken as condition 3.3(3), thus avoiding a reference to the Yosida representation and the reflection is then given an explicit frame-theoretic form. See [4] for details.

4 β (Epicomplete)

E is called epicomplete if $E \xrightarrow{\varphi} \bullet$ monic and epic implies φ an isomorphism. The class of epicomplete objects is denoted EC.

Recall that, for a Tychonoff space X, $B(X)$ denotes the $\mathcal{W}$-object of real-valued Baire functions on X.

We summarize known features of EC, prior to the interpretation of Theorem 2.1 for $\mathcal{R} = EC$.

Theorem 4.1. (*Each item without specific reference can be located in* [11], *with reference to original sources.*)

(a) *$E \in EC$ if and only if E is σ-complete both conditionally, and laterally if and only if $E \approx D(X)$ with X basically disconnected (the X is $\mathcal{Y}E$). Thus, any $B(X) \in EC$.*

(b) ([3]). *$E \approx C(\mathcal{P})$ with $\mathcal{P}$ a P-locale. (Such a $\mathcal{P}$ is the localic intersection of $\{S \mid S$ is dense cozero in $\mathcal{Y}E\}$.)*

(c) *EC is monoreflective, thus the maximum monoreflection. The reflection of A is $\beta A = B(\mathcal{Y}A)/N$, for a certain σ-ideal N.*

EC is H-closed, thus $EC = H\{B(K) \mid K$ compact$\}$.

(d) *If X is Lindelöf and Čech-complete, then $\beta C(X) = B(X)$.*

(e) *For every set I, $\beta F(I) = B(\mathbb{R}^I) = \overset{\omega}{\Uparrow}\{B(\mathbb{R}^J) \mid J \in \mathcal{P}_0(I)\}$.*

We now interpret Theorem 2.1. Most of this is the routine writing-down of items in Theorem 2.1 using information in Theorem 4.1. An exception is Theorem 2.1 (5), which says $A \in H\{B(\mathbb{R}^\omega)\}$. "An" epicompletion of $A \in \mathcal{W}$ is an epic $A \leq E$, with E EC. These are exactly the quotients over A of βA.

Theorem 4.2. *Suppose X is $L\check{C}$ (as is $\mathbb{R}^\omega$).*

(a) *$E \in H\{B(X)\}$ if and only if there is F closed in X such that E is AN epicompletion of $C(F)$.*

(b) *(Note that an F in (a) is again $L\check{C}$.) $C(X)$ has a unique epicompletion if and only if X is discrete and countable (and thus $X \approx \mathbb{N}$, $C(X) \approx C(\mathbb{N})$, is already EC).*

(c) *If X is not countable discrete, there are many epicompletions of $C(X)$.*

Proof. (a) Suppose $E \in H\{B(X)\}$, as $B(X) \overset{\varphi}{\twoheadrightarrow} E$. We have

$$\begin{tikzcd} C(X) \arrow[r, phantom, "\overset{\beta_C}{\leq}"] \arrow[d, two heads, "\varphi_0"'] & \beta C(X) = B(X) \arrow[d, two heads, "\varphi"] \\ \varphi(C(X)) \arrow[r, phantom, "\overset{e}{\leq}"] & E \end{tikzcd} \quad \text{(by Theorem 4.1(d))}$$

where φ_0 is the restriction of φ, e labels the inclusion, and $\varphi\beta_C = e\varphi_0$ (obviously), so e is epic (as a second factor of the epic $\varphi\beta_C$).

By Theorem 3.2, $\varphi(C(X))$ is the desired $C(F)$.

Suppose F is closed in X and $C(F) \overset{e}{\leq} E$ is an epicompletion. We then have

$$\begin{tikzcd} C(X) \arrow[r, phantom, "\overset{\beta_C}{\leq}"] \arrow[d, two heads, "\rho"'] & \beta C(X) \arrow[d, two heads, "\overline{\rho}"] \\ C(F) \arrow[r, phantom, "\overset{e}{\leq}"] & E \end{tikzcd}$$

where ρ is the restriction map described at the beginning of the proof of Theorem 3.2, and then $\exists\overline{\rho}$ with $\overline{\rho}\beta_C = e\rho$ by the universal mapping property of β.

We have $C(F) \overset{i}{\leq} \overline{\rho}(\beta C(X)) \overset{j}{\leq} E$ (i, j are labels) with $ji = e$. But $\overline{\rho}(\beta C(X)) \in EC$ (by Theorem 4.1(c)), and e is epic, thus also j. So j is equality.

(b) If $C(X) \approx C(\mathbb{N})$, already $C(X) \in EC$, so is its only epicompletion.

If $C(X)$ has a unique epicompletion, it must be $C(X) \leq B(X)$ (Theorem 4.1 (d)), and this must be an essential embedding (because any $A \in \mathcal{W}$ has a (unique) essential epicompletion ([2], §9)). If X has a non-void nowhere dense zero-set Z, then the characteristic function $\chi(Z) \in B(X)$, and there is no $0 < a \in C(X)$ with $a \leq \chi(Z)$: $C(X) \leq B(X)$ is not essential. Thus there is no such Z, so X is what is called an almost P-space. But the only almost P-space which is LČ is ($\approx$) $\mathbb{N}$.

(c) See [1] and [2] for several constructions. We omit details. $\square$

Referring to Theorem 4.2, let $\mathcal{ECS}(\mathbb{R}^\omega)$ stand for the family of epicompletions of objects of the form $C(T)$, for T closed in $\mathbb{R}^\omega$.

Summing up, we write down Theorem 2.1 for $\mathcal{W} \xrightarrow{\beta} EC$ using Theorem 4.2 and some of Theorem 4.1.

Corollary 4.3. *For $A \in \mathcal{W}$, the following are equivalent:*

(1) $A \in EC$.

(2) *There is I with a surjection $B(\mathbb{R}^I) \twoheadrightarrow A$.*

(3) $A \in \operatorname{inj}\{F(\omega) \leq B(\mathbb{R}^\omega)\}$.

(4) *Each countable $B \overset{i_B}{\leq} A$, has the property*

$$\text{there is } \beta B \xrightarrow{\bar{i}_B} A \text{ with } \bar{i}_B \beta_B = i_B. \tag{$*$}$$

(5) $A \in \overset{\omega}{\biguplus} \mathcal{ECS}(\mathbb{R}^\omega)$.

(6) $A \in \overset{\omega}{\biguplus} EC$.

The comparison of Corollary 4.3 (4) and (5) with Corollary 3.3 (4) and (5), shows a huge difference between c^3 (or any essential reflection) with β and identifies some special classes of EC objects which might deserve further study. (It is quite rare that any $A \leq \beta A$ is essential; see [2], §9.)

We consider the analogue of Corollary 3.3 (4) for β. Recall that for $B \leq A$, $\beta B \leq A$ means that the $\bar{i}_B$ in 2.8 (4) is one-to-one.

Theorem 4.4. *Suppose $A \in \mathcal{W}$. For every countable $B \leq A$,*

$$\beta B \leq A \text{ if and only if } A \approx \mathbb{R}^n \text{ for some } n \in \mathbb{N}. \tag{$*$}$$

Proof. Notice that either the condition implies $A \in EC$: for $A \in EC$ (or just a vector lattice), $A \approx \mathbb{R}^n$ ($n \in \mathbb{N}$) means $|\mathcal{Y}A| = n$ (and $\mathbb{R}^n = C(\{0, 1, \dots, n-1\})$).

($\Longleftarrow$) We omit the easy proof.

($\Longrightarrow$) We show that $\mathcal{Y}A$ infinite $\Longrightarrow$ A fails $(*)$.

(i) $A = C(\mathbb{N})$ fails $(*)$.

(ii) If $A \in EC$ and $\mathcal{Y}A$ is infinite, then there is an embedding $C(\mathbb{N}) \leq A$.

(iii) If $A \in EC$ and $\mathcal{Y}A$ is infinite, then A fails $(*)$.

For (i): Let $B \leq C(\mathbb{N})$ be generated by rational multiples of the characteristic functions χ_p of the $p \in \mathbb{N}$. A little thought reveals that the uniform completion $uB = c^3B = C(\alpha\mathbb{N})$, where $\alpha\mathbb{N} = \mathbb{N} \cup \{\alpha\}$ the one-point compactification of $\mathbb{N}$. Then $\beta C(\alpha\mathbb{N}) = B(\alpha\mathbb{N})$ (Theorem 4.1 (d)). Then, the $\bar{i}_B$ is not one-to-one: $\bar{i}_B(\psi_\alpha) = 0$.

For (ii): As with any infinite Hausdorff space, there is countable $L = \{x_n\} \subseteq \mathcal{Y}A$ on pairwise disjoint open sets $\{U_n\}$ in $\mathcal{Y}A$ with $U_n \cap L = \{x_n\}$ $\forall n$. We have $L \approx \mathbb{N}$. Since $\mathcal{Y}A$ is basically disconnected, thus zero-dimensional ([8]). The U_n may be chosen clopen, and $\overline{U} \approx \beta\tilde{U}$ (Čech-Stone). Choose any $p_0 \in U$, and retract $\mathcal{Y}A \overset{\rho}{\twoheadrightarrow} \overline{U}$ as $\rho(x) = [x$, if $x \in \overline{U}$; p_0 if $x \notin \overline{U}]$. Since $\overline{U}$ is clopen, ρ is continuous.

Let $f \in C(L)$. Extend to $\overline{f} \in C(U)$ by $f(U_n) = \{f(x_n)\}$. Then extend $\overline{f}$ to $\overline{\overline{f}} \in D(\beta\overline{U})$ (since $\overline{U} = \beta U$). Now $\overline{\overline{f}} \circ \rho \in D(\mathcal{Y}A)$ ($(\overline{\overline{f}} \circ \rho)^{-1}\mathbb{R} = \overline{f}^{-1}(\mathbb{R}) \subseteq \overline{U} - u$, which is nowhere dense). Define $C(L) \overset{\tilde{\rho}}{\leq} D(\mathcal{Y}A) \in A$ as $\tilde{\rho}(f) = \overline{\overline{f}} \circ \rho$. Such compositions preserve ℓ-group operations (and multiplication) and constants, so $\tilde{\rho}(1) = 1$, and $\tilde{\rho} \in \mathcal{W}$.

For (iii): In (ii) we have $C(L) \overset{\tilde{\rho}}{\leq} A$, which we re-name $C(\mathbb{N}) \overset{k}{\leq} A$. In (i), we have countable $B \overset{i_B}{\leq} C(\mathbb{N})$ with $\bar{i}_B$ not 1-1. We have

$$\begin{tikzcd}
B \arrow[r, phantom, "\overset{i_B}{\leq}"] \arrow[d, "\beta_B"'] & c(\mathbb{N}) \arrow[r, phantom, "\overset{k}{\leq}"] & A \\
\beta B \arrow[ur, "\bar{i}_B"] \arrow[urr, "\bar{j}_B"'] & &
\end{tikzcd}$$

with the inclusion $B \overset{j_B}{\leq} A$ being $j_B = ki_B$ and with $\bar{i}_B\beta_B = i_B$, $\bar{j}_B\beta_B = j_B = ki_B$. Thus $\bar{j}_B\beta_B = k\bar{i}_B\beta_B$, so $\bar{j}_B = k\bar{i}_B$ since β_B is epic. Since $\bar{i}_B$ is not one-to-one, neither is $\bar{j}_B$. $\square$

Let $B\mathcal{S}(\mathbb{R}^\omega) \equiv \{B(T) \mid T \text{ dense in } \mathbb{R}^\omega\}$. The analogue of Corollary 3.3(5) for β is the condition

$$A \in \overset{\omega}{\Cup}\{B(T) \mid T \text{ closed in } \mathbb{R}^\omega\}. \tag{$**$}$$

All we have to say is: sometimes this happens, sometimes not.

Remark 4.5. (a) There are A satisfying $(**)$: Obviously, any $B(T)$; less trivially, ([11]) for uncountable I, $B(\mathbb{R}^I) = \overset{\omega}{\Cup}\{B(\mathbb{R}^J) \mid J \in \mathcal{P}_0(I)\}$.

(b) There are many A failing $(**)$. The countable chain condition, ccc, of a space or $\mathcal{W}$-object is relevant here. X (resp., A) has ccc if there is no uncountable pairwise disjoint family of non-void open sets in X (respectively, non-zero positive elements in A). A has ccc if and only if $\mathcal{Y}A$ does (because coz$A$ is a base in $\mathcal{Y}A$).

If A has ccc and satisfies $(**)$, then in the Yosida representation $A \approx D(\mathcal{Y}A)$, each $a \in A$ is locally constant on a dense open subset of $\mathcal{Y}A$. (If $A = \overset{\omega}{\Cup} B(T_\alpha)$, then each $B(T_\alpha)$ has ccc, and it follows that T_α is a copy $\mathbb{N}_\alpha$ of $\mathbb{N}$. For each α, $C(\mathbb{N}_\alpha) = \beta\mathbb{N}_\alpha \overset{\tilde{c}}{\twoheadleftarrow} \mathcal{Y}A)$. If $a \in C(\mathbb{N}_\alpha)$, then a "$=$" $a \circ \tau$ is locally constant on $\tau^{-1}(\mathbb{N}_\alpha)$.)

Consider the absolute (projective cover) $[0,1] \overset{\pi}{\twoheadleftarrow} a[0,1]$. Using irreducibility of π: Since $[0,1]$ has ccc, so do $a[0,1]$, and also $A = D(a[0,1])$. Here $C([0,1]) \leq A$, as $f \mapsto f \circ \pi$. <u>No</u> continuous nonconstant f has $f \circ \pi$ locally constant on a dense subset of $a[0,1]$. Thus A fails $(**)$.

(c) The class EC consists exactly of the $D(X)$, X compact and basically disconnected. The class σBA of σ-complete Boolean algebras consists exactly of the clopen algebras clop$X$ for the same X [16]. So, the various properties of EC's considered here have direct translations to σBA. For example, corresponding to 4.6 are the σBA's of the form $\mathcal{A} \in \overset{\omega}{\Cup}\{\mathcal{B}(T) \mid T \text{ closed in } \mathbb{R}^\omega\}$, $\mathcal{B}$ denoting the σ-field of Baire sets.

We leave the subject for now.

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References

[1] Ball, R.N. and Hager, A.W., *Characterization of epimorphisms in Archimedean lattice-ordered groups and vector lattices*, In: "Lattice-Ordered Groups", Math. Appl. 48, Kluwer Academic Publisher, 1989, 175-205.

[2] Ball, R.N. and Hager, A.W., *Epicompletion of Archimedean ℓ-groups and vector lattices with weak unit*, J. Austral. Math. Soc. Ser. A 48(1) (1990), 25-56.

[3] Ball, R.N., Hager, A.W., and Walters-Wayland, J., *Pointfree pointwise suprema in unital archimedean ℓ-groups*, J. Pure Appl. Algebra 219(11) (2015), 4793-4815.

[4] Banaschewski, B., *On the function ring functor in pointfree topology*, Appl. Categ. Structures 13 (2005), 305-328.

[5] Banaschewski, B., *On the function rings of pointfree topology*, Kyungpook Math. J. 48(2) (2008), 195-206.

[6] Banaschewski, B. *On the characterization of the function rings in pointfree topology*, Lecture in the conference: Aspects of Contemporary Topology V, Vrije Universiteit Brussels, September 2014.

[7] Engelking, R. "General Topology", Revised and completed edition, Sigma Series in Pure Mathematics 6, Heldermann Verlag, 1989.

[8] Gillman, L. and Jerison, M., "Rings of Continuous Functions", Graduate Texts in Mathematics 43, Springer-Verlag, 1976.

[9] Hager, A.W. *Algebraic closures of ℓ-groups of continuous functions*, In: "Rings of Continuous Functions" (C.E. Aull, eds.). Dekker Notes 95, 1985, 165-194.

[10] Hager, A.W. and Madden, J., *Essential reflections versus minimal embeddings*, J. Pure Appl. Algebra 37(1) (1985), 27-32.

[11] Hager, A.W. and Madden, J., *The H-closed monoreflections, implicit operations, and countable composition, in archimedean lattice-ordered groups with weak unit*, Appl. Categ. Structures 24(5) (2016), 605-617.

[12] Hager, A.W. and Robertson, L.C., *Representing and ringifying a Riesz space*, Symposia Mathematica XXI, Academic Press (1977), 411-431.

[13] Henriksen, M., Isbell, J., and Johnson, D., *Residue class fields of lattice-ordered algebras*, Fund. Math. 50 (1961), 107-117.

[14] Isbell, J., "Uniform Spaces", Math. Surveys 12, American Math. Society, 1964.

[15] Pasynkov, B.A., *On the spectral decomposition of topological spaces*, Mat. Sb. 66(108) (1965), 35-79, and Amer. Math. Soc. Translations Series 2 (74) (1968), 87-134.

[16] Sikorski, R. "Boolean Algebras", Third Edition, Springer, 1969.

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