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Pointfree topology version of image of real-valued continuous functions

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Abstract. Let $\mathcal{R}L$ be the ring of real-valued continuous functions on a frame L as the pointfree version of $C(X)$, the ring of all real-valued continuous functions on a topological space X. Since $C_c(X)$ is the largest subring of $C(X)$ whose elements have countable image, this motivates us to present the pointfree version of $C_c(X)$. The main aim of this paper is to present the pointfree version of image of real-valued continuous functions in $\mathcal{R}L$. In particular, we will introduce the pointfree version of the ring $C_c(X)$. We define a relation from $\mathcal{R}L$ into the power set of $\mathbb{R}$, namely *overlap*. Fundamental properties of this relation are studied. The relation overlap is a pointfree version of the relation defined as $\mathrm{Im}(f) \subseteq S$ for every continuous function $f : X \to \mathbb{R}$ and $S \subseteq \mathbb{R}$.

1 Introduction

As is well known, $C(X)$ denotes the ring of all real-valued continuous functions on a topological space X. Undoubtedly, the book *Rings of Continuous Functions* written by Gillman and Jerison is the best reference to study the

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rings of continuous functions [14]. In [13], $C_c(X)$, the subalgebra of $C(X)$, consisting of functions with countable image is studied. It turns out that $C_c(X)$, although not isomorphic to any $C(Y)$ in general, enjoys most of the important properties of $C(X)$. This subalgebra has recently received some attention, see [6, 16–18].

The concept of a frame, or pointfree topology, is a generalization of the classical topology. The ring of real-valued continuous functions on a frame, that is, $\mathcal{R}L$, as the pointfree version of the ring $C(X)$, has been studied prior to 1996 by some authors such as R.N. Ball and A.W. Hager in [1]. A systematic and indepth study of the ring of real continuous functions in pointfree topology was undertaken by B. Banaschewski in 1997 (see [2, 4, 5]). Also, [3, 7, 15, 19] are valuable references on the subject of frames and the ring $\mathcal{R}L$.

In this paper, we introduce the pointfree version of image of real-valued continuous functions in the ring of real-valued continuous functions on a frame, namely, $\mathcal{R}_\lambda L$. In particular, we will have $\mathcal{R}_c L$ as the pointfree version of the ring $C_c(X)$. For this, we use the subsets of $\mathbb{R}$. One may think that we should use the sublocales of the frame $\mathcal{L}(\mathbb{R})$ instead of the subsets of $\mathbb{R}$. In reply, we say that countability image of a continuous function by its very nature deals with number of points of its range, and is not a topological concept. In other words, the countability image of a continuous function does not seem to lend itself to localic interpretation because it is about the number of points in a set.

This paper is organized as follows. In Section 2, we review some basic notions and properties of frames and the pointfree version of the ring of real-valued continuous functions.

In Section 3, we define the concept of *overlap* for $\alpha \in \mathcal{R}L$ (Definition 3.1). To do this, we introduce an onto (quotient) frame map $i : \mathcal{L}(\mathbb{R}) \to \mathfrak{O}S$ given by $i(p, q) = \{s \in S : p < s < q\}$, where $S \subseteq \mathbb{R}$ is taken as a subspace of $\mathbb{R}$ with usual topology and $\mathfrak{O}S$ is the frame of open subsets of S. For every $\alpha \in \mathcal{R}L$ and $S \subseteq \mathbb{R}$, we show that α is an overlap of S if and only if $\breve{\alpha}$ is a frame map, where $\breve{\alpha} : \mathfrak{O}S \to L$ is given by $\breve{\alpha}(U) = \bigvee\{\alpha(v) : v \in \mathcal{L}(\mathbb{R}), i(v) \subseteq U\}$ (see Theorem 3.8). Also, for every continuous function $f : X \to \mathbb{R}$ and $S \subseteq \mathbb{R}$, we show that $f_\tau : \mathcal{L}(\mathbb{R}) \to \mathfrak{O}X$ is an overlap of S if and only if $\mathrm{Im}(f) \subseteq S$ if and only if there exists a continuous function $g : X \to S$ such that $f(x) = g(x)$ for every $x \in X$ (see Proposition 3.11).

In Section 4, we introduce the ring $\mathcal{R}_\lambda L$ as the pointfree version of the image of real-valued continuous functions.

2 Preliminaries

Here, we recall some definitions and results from the literature on frames and the pointfree topology version of the ring of continuous real-valued functions. Our references for frames are [15] and [19].

A *frame* is a complete lattice L in which the distributive law

$$x \wedge \bigvee S = \bigvee \{x \wedge s : s \in S\}$$

holds for all $x \in L$ and $S \subseteq L$. We denote the top element and the bottom element of L by $\top$ and $\bot$, respectively. The frame of open subsets of a topological space X is denoted by $\mathfrak{O}X$.

A *frame homomorphism* (*or frame map*) is a map between frames which preserves finite meets, including the top element, and arbitrary joins, including the bottom element.

An element $p \in L$ is said to be *prime* if $p < \top$ and $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$. A lattice ordered ring A is called an *f-ring*, if $(f \wedge g)h = fh \wedge gh$ for every $f, g \in A$ and every $0 \leq h \in A$.

Recall the contravariant *functor* Σ from **Frm** to the category **Top** of topological spaces which assigns to each frame L its *spectrum* ΣL of prime elements with $\Sigma_a = \{p \in \Sigma L : a \not\leq p\}$ $(a \in L)$ as its open sets.

An element a of a frame L is said to be *completely below* b, written $a \prec\!\!\prec b$, if there exists a sequence $\{c_q\}$, $q \in \mathbb{Q} \cap [0,1]$, where $c_0 = a,\ c_1 = b$, and $c_p \prec c_q$ if $p < q$ where $u \prec v$ means that $u^* \vee v = \top$. A frame L is called *completely regular* if each $a \in L$ is the join of elements completely below it.

Regarding the *frame of reals* $\mathcal{L}(\mathbb{R})$ and the f-ring $\mathcal{R}L$ of *continuous real functions* on L, we use the notations of [4] (see also [2]).

For every pair $(p,q) \in \mathbb{Q}^2$, put

$$\langle p, q\rangle := \{x \in \mathbb{Q} : p < x < q\} \quad \text{and} \quad]\!]p, q[\![:= \{x \in \mathbb{R} : p < x < q\}.$$

Corresponding to every continuous operation $\diamond : \mathbb{Q}^2 \to \mathbb{Q}$ (in particular $+, ., \wedge, \vee$) we have an operation on $\mathcal{R}L$, denoted by the same symbol $\diamond$, defined by

$$\alpha \diamond \beta(p,q) = \bigvee\{\alpha(r,s) \wedge \beta(u,w) : < r,s > \diamond < u,w > \leq < p,q >\},$$

where $< r, s > \diamond < u, w > \leq < p, q >$ means that for each $r < x < s$ and $u < y < w$ we have $p < x \diamond y < q$. For every $r \in \mathbb{R}$, define the constant frame map $\mathbf{r} \in \mathcal{R}L$ by $\mathbf{r}(p, q) = \top$, whenever $p < r < q$, and otherwise $\mathbf{r}(p, q) = \bot$.

Recall that a frame L is called *spatial* if there exists a topological space X such that $L \cong \mathfrak{O}X$. We have the next proposition.

Proposition 2.1. [10] *A frame L is spatial if and only if $\eta : L \to \mathfrak{O}\Sigma L$ by $\eta(a) = \Sigma_a$, for every $a \in L$, is an isomorphism in* **Frm**.

Here we recall the necessary notations, definitions, and results form [9]. Let $a \in L$ and $\alpha \in \mathcal{R}L$. The sets $\{r \in \mathbb{Q} : \alpha(-, r) \leq a\}$ and $\{s \in \mathbb{Q} : \alpha(s, -) \leq a\}$ are denoted by $L(a, \alpha)$ and $U(a, \alpha)$, respectively. For $a \neq \top$ it is obvious that for each $r \in L(a, \alpha)$ and $s \in U(a, \alpha)$, $r \leq s$. In fact, we have

Proposition 2.2. [9] *If $p \in \Sigma L$ and $\alpha \in \mathcal{R}L$, then $(L(p, \alpha), U(p, \alpha))$ is a Dedekind cut for a real number which is denoted by $\widetilde{p}(\alpha)$.*

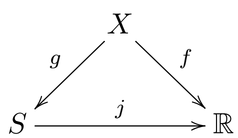
Proposition 2.3. [9] *If p is a prime element of a frame L, then there exists a unique map $\widetilde{p} : \mathcal{R}L \longrightarrow \mathbb{R}$ such that for each $\alpha \in \mathcal{R}L$, $r \in L(p, \alpha)$ and $s \in U(p, \alpha)$ we have $r \leq \widetilde{p}(\alpha) \leq s$.*

Let p be a prime element of L. Throughout this paper, for every $\alpha \in \mathcal{R}L$ we define $\alpha[p] = \widetilde{p}(\alpha)$ (see [11]). For every $\alpha : \mathcal{L}(\mathbb{R}) \to L$, we define $\overline{\alpha} : \Sigma L \to \mathbb{R}$ by $\overline{\alpha}(p) = \alpha[p]$, for $p \in \Sigma L$.

It is well known that the homomorphism $\tau : \mathcal{L}(\mathbb{R}) \to \mathfrak{O}\mathbb{R}$ taking (p, q) to $]\!]p, q[\![$ is an isomorphism (see [4, Proposition 2]).

3 Overlap and its properties

For a topological space X, to say the image of a continuous function $f : X \to \mathbb{R}$ is contained in the set $S \subseteq \mathbb{R}$ is to say there is a morphism $X \xrightarrow{g} S$ in **Top** such that the triangle

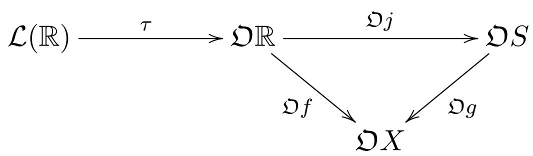


commutes, where j is the inclusion map. Our aim is to extend this notion to pointfree function rings, so that, for instance, we can have an analogue of the $\mathbb{R}$-subalgebra $C_c(X)$ of $C(X)$ whose elements are those functions with countable range.

Regarding the latter, the obvious hurdle is that "countability" is not a topological notion. It is thus not clear how one should define a function $\alpha \in \mathcal{R}L$ to have "countable range". So to obviate this, we, in effect, apply the open-set functor

$$\mathfrak{O} : \mathbf{Top} \to \mathbf{Frm}$$

to the triangle above to obtain the commutative diagram



in **Frm**, after adjoining the morphism $\mathcal{L}(\mathbb{R}) \xrightarrow{\tau} \mathfrak{O}\mathbb{R}$ which maps a generator (p, q) to the open interval $\{x \in \mathbb{R} : p < x < q\}$. Now, starting with an arbitrary $\alpha \in \mathcal{R}L$, we define the concept of "overlapping". We then show that, for any $f \in C(X)$ and $S \subseteq \mathbb{R}$,

$$\mathrm{Im}(f) \subseteq S \quad \Longleftrightarrow \quad \mathfrak{O}f \text{ is an overlap of } S;$$

thus justifying that this is a "correct" extension of the notion of image for pointfree real-valued functions.

In what follows, L, S and $i : \mathcal{L}(\mathbb{R}) \to \mathfrak{O}S$, denote a frame, a subspace of $\mathbb{R}$ with usual topology, and the onto (quotient) frame map, such that for every $p, q \in \mathbb{Q}$, $i(p, q) = \tau(p, q) \cap S$, respectively.

Definition 3.1. For $\alpha \in \mathcal{R}L$ and $S \subseteq \mathbb{R}$, we say that α is an *overlap of* S (denoted by $\alpha \blacktriangleleft S$) if

$$i(u) \subseteq i(v) \text{ implies } \alpha(u) \leq \alpha(v),$$

for every $u, v \in \mathcal{L}(\mathbb{R})$.

Proposition 3.2. *If $\alpha \in \mathcal{R}L$, then it is not an overlap of $\emptyset$.*

Proof. Suppose that $\alpha \blacktriangleleft \emptyset$. Now, we assume that $p, q, r, s \in \mathbb{Q}$, $p < q$ and $r < s$. Since $\tau(p, q) \cap \emptyset = \emptyset = \tau(r, s) \cap \emptyset$, we conclude that $\alpha(p, q) = \alpha(r, s)$.

It follows that $\alpha(p,q) = \bigvee\{\alpha(r,s) : r,s \in \mathbb{Q}\} = \top$. Now, if $p,q,r,s \in \mathbb{Q}$ and $p < q < r < s$, then

$$\bot = \alpha((p,q) \wedge (r,s)) = \alpha(p,q) \wedge \alpha(r,s) = \top,$$

which is a contradiction. □

Definition 3.3. For any $\alpha \in \mathcal{R}L$ and any $S \subseteq \mathbb{R}$, we say that α is a *weakly overlap* of S (denoted by $\alpha \lhd S$) if

$$i(p,q) = i(r,s) \text{ implies } \alpha(p,q) = \alpha(r,s),$$

for every $p,q,r,s \in \mathbb{Q}$.

Example 3.4. Let $\mathrm{Id} : \mathbb{Q} \to \mathbb{R}$ be the identity map. Then $\alpha : \mathfrak{O}\mathbb{R} \to \mathfrak{O}\mathbb{Q}$ is a frame map such that $\alpha(p,q) = \tau(p,q) \cap \mathbb{Q}$. Let $S = \mathbb{R} \setminus \{0\}$. Clearly, $\alpha \lhd S$. Now, if $0 \in \tau(p,q)$ and $p,q \in \mathbb{Q}$, then

$$i(p,q) = \tau(p,q) \cap S \subseteq (\tau(p,0) \cup \tau(0,q)) \cap S = i((p,0) \vee (0,q))$$

and $\alpha(p,q) \not\leq \alpha((p,0) \vee (0,q))$. Thus, α is not an overlap of S.

It is clear that $\alpha \blacktriangleleft S$ implies $\alpha \lhd S$, but the previous example shows that the converse need not hold.

Lemma 3.5. *For any $\alpha \in \mathcal{R}L$ and any $S \subseteq \mathbb{R}$, the following statements are equivalent*:

(1) $\alpha \blacktriangleleft S$.

(2) $i(u) = i(v)$ *implies* $\alpha(u) = \alpha(v)$, *for any* $u, v \in \mathcal{L}(\mathbb{R})$.

(3) $i(p,q) = i(v)$ *implies* $\alpha(p,q) = \alpha(v)$, *for every* $v \in \mathcal{L}(\mathbb{R})$ *and* $p,q \in \mathbb{Q}$.

(4) $i(p,q) \subseteq i(v)$ *implies* $\alpha(p,q) \leq \alpha(v)$, *for any* $v \in \mathcal{L}(\mathbb{R})$ *and any* $p,q \in \mathbb{Q}$.

Proof. (1) ⇒ (2) ⇒ (3) are obviously.

For (3) ⇒ (4), suppose that $i(p,q) \subseteq i(v)$. So

$$i(p,q) = i(p,q) \cap i(v) = i((p,q) \wedge v).$$

By (3), $\alpha(p,q) = \alpha((p,q) \wedge v)$, and hence $\alpha(p,q) \leq \alpha(v)$.

Finally, to show (4) $\Rightarrow$ (1), let $u, v \in \mathcal{L}(\mathbb{R})$ such that $i(u) \subseteq i(v)$. Let $(p,q) \leq u$ where $p, q \in \mathbb{Q}$. Hence $i(p,q) \subseteq i(u) \subseteq i(v)$, so, by (4), $\alpha(p,q) \leq \alpha(v)$. Therefore,

$$\alpha(u) = \alpha(\bigvee_{(p,q)\leq u} (p,q)) = \bigvee_{(p,q)\leq u} \alpha(p,q) \leq \alpha(v).$$

□

Definition 3.6. For $\alpha \in \mathcal{R}L$ and $S \subseteq \mathbb{R}$, define $\breve{\alpha} : \mathfrak{O}S \to L$ by

$$\breve{\alpha}(U) = \bigvee\{\alpha(v) : v \in \mathcal{L}(\mathbb{R}), i(v) \subseteq U\}.$$

It is clear that $\breve{\alpha}(U) = \bigvee\{\alpha(p,q) : \tau(p,q) \cap S \subseteq U\}$.

Lemma 3.7. *For $\alpha \in \mathcal{R}L$ and $S \subseteq \mathbb{R}$,*

(1) *$\breve{\alpha}$ is an order preserving map such that for every $u \in \mathcal{L}(\mathbb{R})$, $\alpha(u) \leq \breve{\alpha}(i(u))$.*

(2) *$\breve{\alpha}i = \alpha$ if and only if $\alpha \blacktriangleleft S$.*

Proof. (1) is clear.

To show (2), first suppose that $\breve{\alpha}i = \alpha$ and $i(u) \subseteq i(v)$. So

$$\alpha(u) = \breve{\alpha}i(u) \leq \breve{\alpha}i(v) = \alpha(v).$$

Therefore, $\alpha \blacktriangleleft S$. Conversely, suppose that $\alpha \blacktriangleleft S$. Let $u \in \mathcal{L}(\mathbb{R})$. We have

$$\begin{aligned} \breve{\alpha}(i(u)) &= \bigvee\{\alpha(v) : v \in \mathcal{L}(\mathbb{R}), i(v) \subseteq i(u)\} \\ &\leq \bigvee\{\alpha(v) : v \in \mathcal{L}(\mathbb{R}), \alpha(v) \leq \alpha(u)\} \\ &= \alpha(u). \end{aligned}$$

So, by (1), $\breve{\alpha}i = \alpha$. □

In the proof of one of the implications in the upcoming theorem we will use the fact that if M is a regular frame and $f, g : M \to L$ are frame maps such that $f(x) \leq g(x)$ for all $x \in M$, then $f = g$.

Theorem 3.8. *For any $\alpha \in \mathcal{R}L$ and any $S \subseteq \mathbb{R}$, the following statements are equivalent:*

(1) $\alpha \blacktriangleleft S$.

(2) $\breve{\alpha}i = \alpha$.

(3) *$\breve{\alpha}$ is a frame map.*

Proof. (1) $\Leftrightarrow$ (2). It follows from Lemma 3.7.

(2) $\Rightarrow$ (3). This is because, $i : \mathcal{L}(\mathbb{R}) \to \mathfrak{O}S$ is an onto frame map and $\breve{\alpha}$ is a well-defined function.

Finally, to see (3) $\Rightarrow$ (2), note that for every $u \in \mathcal{L}(\mathbb{R})$, by Lemma 3.7(1), $(\breve{\alpha}i)(u) \geq \alpha(u)$. Since $\mathcal{L}(\mathbb{R})$ is a regular frame and $\breve{\alpha}i, \alpha : \mathcal{L}(\mathbb{R}) \to L$ are two frame maps, we conclude that $\breve{\alpha}i = \alpha$. □

Corollary 3.9. *For any $\alpha \in \mathcal{R}L$ and any $S \subseteq \mathbb{R}$, the following statements are equivalent*:

(1) $\alpha \blacktriangleleft S$.

(2) *For every* $\{(p_i, q_i)\}_{i\in I}, \{(r_j, s_j)\}_{j\in J} \subseteq \mathbb{Q} \times \mathbb{Q}$, *if*

$$\bigcup_{i\in I} \tau(p_i, q_i) \cap S = \bigcup_{j\in J} \tau(r_j, s_j) \cap S,$$

then $\bigvee_{i\in I} \alpha(p_i, q_i) = \bigvee_{j\in J} \alpha(r_j, s_j)$.

(3) *There exists a unique frame map* $\beta : \mathfrak{O}S \to L$ *such that* $\beta i = \alpha$.

Proof. By Theorem 3.8, it is evident. □

In what follows, for $f \in C(X)$, the frame map

$$f^{-1} \circ \tau : \mathcal{L}(\mathbb{R}) \to \mathfrak{O}X$$

is denoted by f_τ. Note that for $p < q$ in $\mathbb{Q}$,

$$f_\tau(p, q) = \{x \in X : p < f(x) < q\}.$$

Lemma 3.10. *For every $f \in C(X)$, if $Im(f) \subseteq S \subseteq \mathbb{R}$, then $f_\tau \blacktriangleleft S$.*

Proof. Let $p, q \in \mathbb{Q}$ and $u \in \mathcal{L}(\mathbb{R})$. If $\tau(p, q) \cap S \subseteq i(u)$, then

$$\begin{aligned} x \in f_\tau(p, q) \quad &\Rightarrow \quad f(x) \in \tau(p, q) \cap \mathrm{Im}(f) \subseteq \tau(u) \cap S \cap \mathrm{Im}(f) \\ &\Rightarrow \quad x \in f_\tau(u). \end{aligned}$$

Therefore, $f_\tau \blacktriangleleft S$. □

Proposition 3.11. *Let $S \subseteq \mathbb{R}$ and $f \in C(X)$. Then the following statements are equivalent*:

(1) $f_\tau \blacktriangleleft S$.

(2) *There exists a continuous function $g : X \to S$ such that $f(x) = g(x)$, for every $x \in X$.*

(3) $Im(f) \subseteq S$.

Proof. (1) $\Rightarrow$ (3). Suppose that $\mathrm{Im}(f) \nsubseteq S$. Then there exists $x \in X$ such that $y = f(x) \in \mathrm{Im}(f) \setminus S$. Let $p, q \in \mathbb{Q}$ and $p < y < q$. There exist sequences $\{p_n\}_{n\in\mathbb{N}}, \{q_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Q}$ such that $p_n \longrightarrow y$, $q_n \longrightarrow y$ and for every $n \in \mathbb{N}$, $p < p_n < y < q_n < q$. Hence

$$\tau(p,q) \cap S = \bigcup_{n\in\mathbb{N}} (\tau(p,p_n) \cup \tau(q_n,q)) \cap S.$$

By Corollary 3.9, $x \in f_\tau(p,q) = \bigvee_{n\in\mathbb{N}}(f_\tau(p,p_n) \cup f_\tau(q_n,q))$ and it follows that there is $n \in \mathbb{N}$ such that $x \in f_\tau(p,p_n) \cup f_\tau(q_n,q)$, which is a contradiction.

(3) $\Rightarrow$ (1). By Lemma 3.10, it is clear.

(3) $\Leftrightarrow$ (2). It is evident. □

Lemma 3.12. *Let p be a prime element of L. For $\alpha \in \mathcal{R}L$ and $t \in \mathbb{R}$, $\alpha[p] \neq t$ if and only if $\bigvee\{\alpha(-,r) \vee \alpha(s,-) : r,s \in \mathbb{Q}, r < t < s\} \nleq p$.*

Proof. Suppose that $\alpha[p] \neq t$, assume that $\alpha[p] > t$. Hence, there is a rational number r such that $\alpha[p] > r > t$. Thus, by [9, Lemma 3.1], $r \in L(p,\alpha)$, and so, by the definition of $L(p,\alpha)$, $\alpha(-,r) \leq p$. Now, if

$$\bigvee\{\alpha(-,r) \vee \alpha(s,-) : r,s \in \mathbb{Q}, r < t < s\} \leq p,$$

we have

$$\top = \alpha(-,r) \vee \bigvee\{\alpha(-,r) \vee \alpha(s,-) : r,s \in \mathbb{Q}, r < t < s\} \leq p \vee p = p,$$

which contradicts p being a prime element. Therefore,

$$\bigvee\{\alpha(-,r) \vee \alpha(s,-) : r,s \in \mathbb{Q}\} \nleq p.$$

The case $\alpha[p] < t$ is proved similarly.

Conversely, suppose that $\alpha[p] = t$. So, by [9, Lemma 3.1], for every two rationals $r < t < s$, we have $r \in L(p,\alpha)$ and $s \in U(p,\alpha)$. Hence $\alpha(-,r) \vee \alpha(s,-) \leq p$, by the definition of $L(p,\alpha)$ and $U(p,\alpha)$. Thus,

$$\bigvee\{\alpha(-,r) \vee \alpha(s,-) : r,s \in \mathbb{Q}, r < t < s\} \leq p,$$

which contradicts the assumption. □

Proposition 3.13. *For every $\alpha \in \mathcal{R}L$ and $S \subseteq \mathbb{R}$, if $\alpha \blacktriangleleft S$, then $Im(\overline{\alpha}) \subseteq S$.*

Proof. Suppose that $\mathrm{Im}(\overline{\alpha}) \nsubseteq S$. Then there exists $p \in \Sigma L$ such that $\overline{\alpha}(p) = t \in \mathrm{Im}(\overline{\alpha}) \setminus S$. By Lemma 3.12,

$$\bigvee \{\alpha(-,r) \vee \alpha(s,-) : r,s \in \mathbb{Q}, r < t < s\} \leq p.$$

Since $t \notin S$, we conclude that

$$\bigcup \{\tau(r,s) \cap S : r,s \in \mathbb{Q}\} = S = \bigcup \{\tau(-,r) \cap S \vee \tau(s,-) \cap S : r,s \in \mathbb{Q}, r < t < s\}.$$

By Corollary 3.9,

$$\top = \bigvee \{\alpha(r,s) : r,s \in \mathbb{Q}\} = \bigvee \{\alpha(-,r) \vee \alpha(s,-) : r,s \in \mathbb{Q}, r < t < s\} \leq p,$$

which is a contradiction. □

Corollary 3.14. *For any $t \in \mathbb{R}$, the following statements are equivalent:*
(1) *$t \in S$.*
(2) *$\boldsymbol{t} \blacktriangleleft S$, where $\boldsymbol{t} \in \mathcal{R}L$.*

Proof. (1) ⇒ (2). Let $t \in S$ and $u, v \in \mathcal{L}(\mathbb{R})$ with $i(u) \subseteq i(v)$. If $t \in i(u)$, then $\mathbf{t}(u) = \mathbf{t}(v) = \top$ and if $t \notin i(u)$, then $\mathbf{t}(u) = \bot$. Therefore, $\mathbf{t}(u) \leq \mathbf{t}(v)$, which gives that $\mathbf{t} \blacktriangleleft S$.

(2) ⇒ (1). Suppose that $\mathbf{t} \blacktriangleleft S$. So, by Proposition 3.13, $\mathrm{Im}(\overline{\mathbf{t}}) = \{t\} \subseteq S$, that is, $t \in S$. □

Lemma 3.15. *Let L be a spatial frame. For any $\alpha \in \mathcal{R}L$ and the frame isomorphism $\eta : L \to \mathfrak{O}(\Sigma L)$ by $\eta(a) = \Sigma_a$, we have $\eta\alpha = \overline{\alpha}_\tau$.*

Proof. Let $(p,q) \in \mathcal{L}(\mathbb{R})$. We have

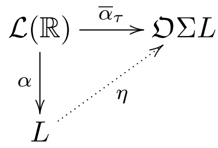
$$\eta\alpha(p,q) = \eta(\alpha(p,q)) = \Sigma_{\alpha(p,q)} = \{x \in \Sigma L : \alpha(p,q) \nleq x\}$$

and $\overline{\alpha}_\tau(p,q) = \{x \in \Sigma L : p < \overline{\alpha}(x) < q\}$. We show that

$$\Sigma_{\alpha(p,q)} = \{x \in \Sigma L : p < \alpha[x] < q\}.$$

Let $x \in \Sigma_{\alpha(p,q)}$, then $\alpha(p,q) \nleq x$. So $\alpha(-,p) \leq x$ and $\alpha(q,-) \leq x$, because x is prime and $\alpha(p,q) \wedge \alpha(-,p) = \bot \leq x$ and $\alpha(p,q) \wedge \alpha(q,-) = \bot \leq x$.

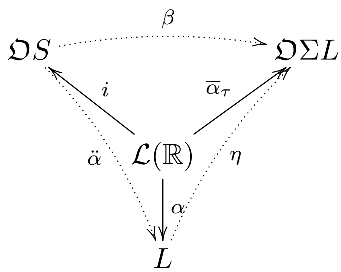
So $p \in L(x, \alpha)$ and $q \in U(x, \alpha)$. Hence $p < \alpha[x] < q$. Thus $x \in \overline{\alpha}_\tau(p, q)$. Therefore, $\eta\alpha(p, q) \leq \overline{\alpha}_\tau(p, q)$ for all $p, q \in \mathbb{Q}$. Hence $\eta\alpha = \overline{\alpha}_\tau$, by the regularity of $\mathcal{L}(\mathbb{R})$. Consequently, $\eta\alpha = \overline{\alpha}_\tau$ and the following diagram is commutative:



□

Proposition 3.16. *Let L be a spatial frame. Then the converse of the Proposition* 3.13 *holds.*

Proof. Let L be a spatial frame and $\mathrm{Im}(\overline{\alpha}) \subseteq S$. Then, by Proposition 3.11, $\overline{\alpha}_\tau \blacktriangleleft S$. Now, by Corollary 3.9, there exists a unique frame map $\beta : \mathfrak{O}S \to \mathfrak{O}\Sigma L$ such that $\beta i = \overline{\alpha}_\tau$. Also, since L is spatial, we have the isomorphism $\eta : L \to \mathfrak{O}\Sigma L$ with $\eta(a) = \Sigma_a$. Now, define $\ddot{\alpha} : \mathfrak{O}S \to L$ by $\ddot{\alpha} = \eta^{-1}\beta$. See the following diagram:



By Corollary 3.9, it is sufficient to show that $\ddot{\alpha} i$ is a unique frame map such that $\ddot{\alpha} i = \alpha$. To do this, let $(p, q) \in \mathcal{L}(\mathbb{R})$. So, by Lemma 3.15, we have

$$\begin{aligned} \ddot{\alpha} i(p, q) &= \ddot{\alpha}(i(p, q)) \\ &= \eta^{-1}\beta(i(p, q)) \\ &= \eta^{-1}(\beta i)(p, q) \\ &= \eta^{-1}\overline{\alpha}_\tau(p, q) \\ &= \alpha(p, q). \end{aligned}$$

Also, since the frame map β is unique, it follows that $\ddot{\alpha}$ is unique. □

Remark 3.17. Recall from [8] that for an infinite cardinal number k, then X is a (Tychonoff) space of weight at most k. This means that X has a basis for its topology of cardinality at most k. Moreover, let $\mathcal{I}$ be a k^+-complete ideal of subsets of X. This means that $\mathcal{I}$ is an ideal of subsets of X which has the following property: if $\mathcal{A} \subseteq \mathcal{I}$ and $|\mathcal{A}| \leq k$, then $\bigcup \mathcal{A} \in \mathcal{I}$. Now, let $L = \mathfrak{O}X$. We define a relation $\sqsubseteq$ on L as follows: for $U, V \in L$ we put

$$U \sqsubseteq V \quad \text{if and only if} \quad U \setminus V \in \mathcal{I}.$$

Next, an equivalence relation $\sim$ on L is defined by

$$U \sim V \quad \text{if and only if} \quad U \sqsubseteq V \text{ and } V \sqsubseteq U.$$

For $U \in L$, we let $[U]$ denote its $\sim$-equivalence class. Now, put $M = L/\sim$, and define a partial order $\leq$ on M by

$$[U] \leq [V] \quad \text{if and only if} \quad U \sqsubseteq V.$$

This definition is well defined and M is a completely regular frame with bottom $[\emptyset] = \{U \in \mathfrak{O}X : U \in \mathcal{I}\}$ and top $[X] = \{U \in \mathfrak{O}X : X \setminus U \in \mathcal{I}\}$. For more details see [8].

Let $\alpha \in \mathcal{R}L$ and $\{S_i : i \in I\}$ be a family of subsets of $\mathbb{R}$. In the following example, we show that if $\alpha \blacktriangleleft S_i$, for all $i \in I$, then α may not be an overlap of $\bigcap \{S_i : i \in I\}$.

Example 3.18. Consider $X = [0, 1]$ and $k = \aleph_0$. Let

$$\mathcal{I} = \{A \subseteq [0, 1] : \text{the measure of } A \text{ is zero}\}.$$

It is clear that $\mathcal{I}$ is a k^+-complete ideal of subsets of X. Now, let $\alpha : X \to \mathbb{R}$ be defined by $\alpha(x) = x$. Consider the frame map $\alpha_\tau : \mathcal{L}(\mathbb{R}) \to \mathfrak{O}X$ defined by $\alpha_\tau(p, q) = \tau(p, q) \cap [0, 1]$. Now, let $L = \mathfrak{O}X$ and put $M = L/\sim$, where $\sim$ is the equivalence relation on L defined in Remark 3.17. Define $\beta : \mathcal{L}(\mathbb{R}) \to M$ by

$$\beta(u) = [\alpha_\tau(u)] = [\tau(u) \cap [0, 1]].$$

Let c be an arbitrary element of $\mathcal{I}$. Let $S_c = [0, 1] \setminus c$. We claim that $\beta \blacktriangleleft S_c$. Let $u, v \in \mathcal{L}(\mathbb{R})$ and $i(u) \subseteq i(v)$. Then

$$\tau(u) \cap [0, 1] \cap S_c \subseteq \tau(v) \cap [0, 1] \cap S_c,$$

which follows that

$$\tau(u) \cap [0,1] \setminus \tau(v) \cap [0,1] \subseteq c.$$

Since $c \in \mathcal{I}$, then

$$(\tau(u) \cap [0,1]) \setminus (\tau(v) \cap [0,1]) \in \mathcal{I}.$$

Hence, by Remark 3.17,

$$\tau(u) \cap [0,1] \sqsubseteq \tau(v) \cap [0,1],$$

which follows that

$$[\tau(u) \cap [0,1]] \leq [\tau(v) \cap [0,1]].$$

Therefore, $\beta(u) \leq \beta(v)$. Thus, $\beta \blacktriangleleft S_c$. Also, we have $\bigcap_{c\in\mathcal{I}} S_c = \emptyset$. Hence, by Proposition 3.2, β is not an overlap of $\bigcap\{S_c : c \in \mathcal{I}\} = \emptyset$.

Proposition 3.19. *Let $\alpha : \mathcal{L}(\mathbb{R}) \to L$ and $\beta : L \to M$ be frame maps.*
(1) *If $\alpha \blacktriangleleft S$ then $\beta \circ \alpha \blacktriangleleft S$.*
(2) *If β is a monomorphism and $\beta \circ \alpha \blacktriangleleft S$, then $\alpha \blacktriangleleft S$.*

Proof. (1) Let $u, v \in \mathcal{L}(\mathbb{R})$ and $i(u) \subseteq i(v)$, then $\alpha(u) \leq \alpha(v)$. Therefore, $\beta \circ \alpha(u) \leq \beta \circ \alpha(v)$. Hence $\beta \circ \alpha \blacktriangleleft S$.
(2) Let $u, v \in \mathcal{L}(\mathbb{R})$ and $i(u) = i(v)$, then $\beta \circ \alpha(u) = \beta \circ \alpha(v)$. Since β is a monomorphism, $\alpha(u) = \alpha(v)$. □

Remark 3.20. In Proposition 3.19 (2), the condition that β is a monomorphism is necessary.

Example 3.21. In Example 3.18, for every $c \in \mathcal{I}$, $\beta \blacktriangleleft S_c = [0,1] \setminus c$, but α_τ is not an overlap of $S_c = [0,1] \setminus c$, because $\mathrm{Im}(\alpha) = [0,1]$.

4 The ring $\mathcal{R}_\lambda L$

Let S_1 and S_2 be subsets of $\mathbb{R}$. For the binary operations $\diamond = +, \cdot, \wedge, \vee : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, we define

$$S_1 \diamond S_2 = \{a \diamond b : a \in S_1, b \in S_2\}.$$

Lemma 4.1. *Let S_1 and S_2 be subsets of $\mathbb{R}$ and $S_\diamond = S_1 \diamond S_2$, for any $\diamond \in \{+, \cdot, \wedge, \vee\}$. Let $r, s \in \mathbb{Q}$, $u \in \mathcal{L}(\mathbb{R})$ and $\diamond \in \{+, \cdot, \wedge, \vee\}$. If $\tau(r,s) \cap S_\diamond \subseteq \tau(u) \cap S_\diamond$, then*

$$A_i := \bigcup\{\tau(p,q) \cap S_i : p, q \in \mathbb{Q}\,,\ \tau(p,q) \diamond \tau(t,k) \subseteq \tau(r,s),\ \textit{for some } t, k \in \mathbb{Q}\}$$

is a subset of

$$B_i := \bigcup\{\tau(a,b) \cap S_i : a, b \in \mathbb{Q}\,,\ \tau(a,b) \diamond \tau(c,d) \subseteq \tau(u),\ \textit{for some } c, d \in \mathbb{Q}\},$$

for $i = 1, 2$.

Proof. Let $x \in A_1$. Then there exist $p, q, t, k \in \mathbb{Q}$ such that $x \in \tau(p,q) \cap S_1$ and $\tau(p,q) \diamond \tau(t,k) \subseteq \tau(r,s)$. Hence for every $y \in \tau(t,k) \cap S_2$, $x \diamond y \in \tau(r,s) \cap S_\diamond$. Thus, there exist sequences

$$\{p_n\}_{n\in\mathbb{N}}, \{q_n\}_{n\in\mathbb{N}}, \{t_n\}_{n\in\mathbb{N}}, \{k_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Q}$$

such that $p_n, q_n \longrightarrow x$, $t_n, k_n \longrightarrow y$ and for every $n \in \mathbb{N}$,

$$p < p_n < p_{n+1} < x < q_{n+1} < q_n < q \text{ and}$$
$$t < t_n < t_{n+1} < y < k_{n+1} < k_n < k.$$

Since $x \diamond y \in \tau(u)$, $p_n \diamond t_n \longrightarrow x \diamond y$ and $q_n \diamond k_n \longrightarrow x \diamond y$, we conclude that there exists $n \in \mathbb{N}$ such that

$$x \diamond y \in \tau(p_n, q_n) \diamond \tau(t_n, k_n) \subseteq \tau(u)$$

and $x \in \tau(p_n, q_n) \cap S_1$, which shows that $x \in B_1$. The case for $i = 2$ is proved similarly. □

Proposition 4.2. *Let S_1 and S_2 be subsets of $\mathbb{R}$. If $\alpha, \beta \in \mathcal{R}L$ such that $\alpha \blacktriangleleft S_1$ and $\beta \blacktriangleleft S_2$, then $\alpha \diamond \beta \blacktriangleleft S_1 \diamond S_2$, where $\diamond = +, \cdot, \wedge, \vee$.*

Proof. Let $S_\diamond = S_1 \diamond S_2$, $r, s \in \mathbb{Q}$ and $u \in \mathcal{L}(\mathbb{R})$. If $\tau(r,s) \cap S_\diamond \subseteq \tau(u) \cap S_\diamond$, then, by Lemma 4.1, we have

$$\begin{aligned} \alpha \diamond \beta(r,s) &= \bigvee\{\alpha(p,q) \wedge \beta(t,k) : < p,q > \diamond < t,k > \subseteq < r,s >\} \\ &\leq \bigvee\{\alpha(a,b) \wedge \beta(c,d) : < a,b > \diamond < c,d > \subseteq \tau(u)\} \\ &= \alpha \diamond \beta(u). \end{aligned}$$

Therefore, $\alpha \diamond \beta \blacktriangleleft S_\diamond$. □

Definition 4.3. Let λ be an infinite cardinal number and $\alpha \in \mathcal{R}L$. We say that α *has the pointfree λ-image* if there exists a subset $S \subseteq \mathbb{R}$ such that $|S| < \lambda$ and $\alpha \blacktriangleleft S$.

Corollary 4.4. *For every $\alpha \in \mathcal{R}L$ and $S \subseteq \mathbb{R}$, if $\lambda < \aleph_1$ (the first uncountable cardinal) and α has the pointfree λ-image, then $Im(\overline{\alpha})$ is countable.*

Proof. It follows from Proposition 3.13. □

Corollary 4.5. *Let $f \in C(X)$, then the following statements are equivalent*:

(1) *The frame map f_τ has the pointfree λ-image.*

(2) *$Im(f)$ is a subset of $\mathbb{R}$ with $|Im(f)| < \lambda$.*

Proof. It follows from Lemma 3.10 and Proposition 3.11. □

Remark 4.6. Let L be a frame such that $\Sigma L = \emptyset$. For every $\alpha \in \mathcal{R}L$, we have $\mathrm{Im}(\overline{\alpha}) = \emptyset$. By Proposition 3.2, countability of $\mathrm{Im}(\overline{\alpha})$ does not imply countability of pointfree image of α.

Definition 4.7. For every frame L, we put

$$\mathcal{R}_\lambda L = \{\alpha \in \mathcal{R}L : \alpha \text{ has the pointfree } \lambda\text{-image}\}.$$

For every $r \in \mathbb{R}$, if $S_r = \{r\}$, then $\mathbf{r} \blacktriangleleft S_r$. Therefore,

$$\{\mathbf{r} : r \in \mathbb{R}\} \subseteq \mathcal{R}_\lambda L.$$

Remark 4.8. If $\lambda > \aleph_1$, then $\mathcal{R}_\lambda L = \mathcal{R}L$, because for every $\alpha \in \mathcal{R}L$, $\alpha \blacktriangleleft \mathbb{R}$.

Corollary 4.9. *Let L be a frame. Then the set $\mathcal{R}_\lambda L$ is a sub-f-ring of $\mathcal{R}L$.*

Proof. By Proposition 4.2, it is evident. □

Remark 4.10. We have

$$\mathcal{R}_c L := \{\alpha \in \mathcal{R}L : \text{there exists a countable subset } S \text{ such that } \alpha \blacktriangleleft S\}$$

as the pointfree version of the ring $C_c(X)$, the subalgebra of $C(X)$, consisting of functions with countable image.

A study of z_c-ideals and prime ideals in the ring $\mathcal{R}_c L$ is done in [12].

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References

[1] Ball, R.N. and Hager, A.W., *On the localic Yoshida representation of an archimedean lattice ordered group with weak order unit*, J. Pure Appl. Algebra, 70 (1991), 17-43.

[2] Ball, R.N. and Walters-Wayland, J., *$C-$ and C^*- quotients on pointfree topology*, Dissertations Mathematicae (Rozprawy Mat), 412 Warszawa (2002), 62 pp.

[3] Banaschewski, B., *Pointfree topology and the spectra of f-rings*, Ordered algebraic structures, (Curacao 1995), Kluwer Acad. Publ. (1997), 123-148.

[4] Banaschewski, B., *The real numbers in pointfree topology*, Textos Mat. Sér. B 12, University of Coimbra, 1997.

[5] Banaschewski, B. and Gilmour, C.R.A., *Pseudocompactness and the cozero part of a frame*, Comment. Math. Univ. Carolin. 37 (1996), 577-587.

[6] Bhattacharjee, P., Knox, M.L., and McGovern, W.W., *The classical ring of quotients of $C_c(X)$*, Appl. Gen. Topol. 15(2) (2014), 147-154.

[7] Dube, T. and Ighedo, O., *On z-ideals of pointfree function rings*, Bull. Iran. Math. Soc. 40 (2014), 657-675.

[8] Dube, T., Iliadis, S., Van Mill, J., and Naidoo, I., *A Pseudocompact completely regular frame which is not spatial*, Order 31(1) (2014), 115-120.

[9] Ebrahimi, M.M. and Karimi Feizabadi, A., *Pointfree prime representation of real Riesz maps*, Algebra Universalis 54 (2005), 291-299.

[10] Ebrahimi, M.M. and Mahmoudi, M., "Frames", Technical Report, Department of Mathematics, Shahid Beheshti University, 1996.

[11] Estaji, A.A., Karimi Feizabadi, A., and Abedi, M., *Zero sets in pointfree topology and strongly z-ideals*, Bull. Iran. Math. Soc 41(5) (2015), 1071-1084.

[12] Estaji, A.A., Karimi Feizabadi, A., and Robat Sarpoushi, M., *z_c-Ideals and prime ideals in the ring $\mathcal{R}_cL$*, to appear in Filomat.

[13] Ghadermazi, M., Karamzadeh, O.A.S., and Namdari, M., *On the functionally countable subalgebra of $C(X)$*, Rend. Sem. Mat. Univ. Padova 129 (2013), 47-69.

[14] Gillman, L. and Jerison, M., "Rings of continuous functions", Springer-Verlag, 1976.

[15] Johnstone, P.T., "Stone spaces", Cambridge Univ. Press, 1982.

[16] Karamzadeh, O.A.S., Namdari, M., and Soltanpour, *On the locally functionally countable subalgebra of $C(X)$*, Appl. Gen. Topol. 16 (2015), 183-207.

[17] Karamzadeh, O.A.S. and Rostami, M., *On the intrinsic topology and some related ideals of $C(X)$*, Proc. Amer. Math. Soc. 93 (1985), 179-184.

[18] Namdari, M. and Veisi, A., *Rings of quotients of the subalgebra of $C(X)$ consisting of functions with countable image*, Inter. Math. Forum 7 (2012), 561-571.

[19] Picado, J. and Pultr, A., "Frames and Locales: topology without points", Birkhäuser/Springer, Basel AG, 2012.

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