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On property (A) and the socle of the f-ring $Frm(\mathcal{P}(\mathbb{R}), L)$

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Abstract. For a frame L, consider the f-ring $\mathcal{F}_{\mathcal{P}}L = Frm(\mathcal{P}(\mathbb{R}), L)$. In this paper, first we show that each minimal ideal of $\mathcal{F}_{\mathcal{P}}L$ is a principal ideal generated by f_a, where a is an atom of L. Then we show that if L is an $\mathcal{F}_{\mathcal{P}}$-completely regular frame, then the socle of $\mathcal{F}_{\mathcal{P}}L$ consists of those f for which $coz(f)$ is a join of finitely many atoms. Also it is shown that not only $\mathcal{F}_{\mathcal{P}}L$ has Property (A) but also if L has a finite number of atoms then the residue class ring $\mathcal{F}_{\mathcal{P}}L/\text{Soc}(\mathcal{F}_{\mathcal{P}}L)$ has Property (A).

1 Introduction

The *socle* of a ring R, denoted by $\text{Soc}(R)$, is the ideal generated by the minimal ideals of R. In [19], the authors showed that for a completely regular Hausdorff space X, the socle of the ring $C(X)$, which is denoted by $C_F(X)$, is the ideal consisting of all functions which are zero everywhere except on a finite number of points. In [8] it is shown that X is a P-space if and only if $C(X)$ is an $\aleph_0$-selfinjective ring or, equivalently, if and only if $C(X)/C_F(X)$

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is $\aleph_0$-selfinjective (also see [1]). In [6, 7] the authors showed that the socle of $\mathcal{R}L$, the ring of real-valued continuous functions on a completely regular frame L, is the ideal consisting of functions whose cozero elements are finite joins of atoms of L which is a pointfree version of $C(X)$.

Let L be a frame and $F(L) := Frm(\mathcal{L}(\mathbb{R}), \mathcal{S}L)$, where $\mathcal{S}L$ is the dual of the co-frame of all sublocales of L. In [10], they showed that the lattice ordered ring $F(L)$ is a pointfree counterpart of the ring R^X with X a topological space (also see [9, 11]). They thus have a pointfree analogue of the concept of an arbitrary, not necessarily (semi) continuous, real function on L. In [25], they showed that $F(L) = C(\mathcal{S}_{\mathfrak{c}}(L))$ is always order complete, where $\mathcal{S}_{\mathfrak{c}}(L)$ is a frame of closedly generated sublocales. Also, Karimi Feizabadi et al. in [18] showed that $\mathcal{F}_{\mathcal{P}}L := Frm(\mathcal{P}(\mathbb{R}), L)$ is an f-ring, as a generalization of all functions from a set X into $\mathbb{R}$, because $R^X \cong \mathcal{F}_{\mathcal{P}}(\mathcal{P}(\mathbb{X}))$. Also, they showed that $\mathcal{F}_{\mathcal{P}}L$ is isomorphic to a sub-f-ring of $\mathcal{R}(L)$, the ring of real-valued continuous functions on L.

One of the important properties of commutative Noetherian rings is that the annihilator of an ideal I consisting entirely of zero-divisors is nonzero [17, p.56]. However this result fails for some non-Noetherian ring, even if the ideal I is finitely generated [17, p.63]. Huckaba and Keller [15] introduced the following: A commutative ring R has *Property* (A) if every finitely generated ideal of R consisting entirely of zero-divisors has a nonzero annihilator. Property (A) was originally studied by Quentel [26]. The class of commutative rings with property (A) is quite large and has been studied by many authors [2, 12, 14, 15]. Polynomial rings, rings whose classical ring of quotionts is von Neumann regular, Noetherian rings [17, p. 56], and rings whose prime ideals are maximal [12] are well known examples of rings in this class. In [13], Hong et al. extend Property(A) to non-commutative rings as follows: A ring R has right (left) *Property* (A) if every finitely generated tow-sided ideal of R consisting entirely of left (right) zero-divisors has a right (left) non-zero annihilator. A ring R is said to have Property (A) if R has right and left Property (A).

In this paper, for the f-ring $\mathcal{F}_{\mathcal{P}}L$, first we show that each minimal ideal of $\mathcal{F}_{\mathcal{P}}L$ is a principal ideal generated by f_a, where a is an atom of frame L. Then we show that if L is an $\mathcal{F}_{\mathcal{P}}$-completely regular frame, then the socle of $\mathcal{F}_{\mathcal{P}}L$ consists of those f for which $coz(f)$ is a join of finitely many atoms. Also it is shown that not only $\mathcal{F}_{\mathcal{P}}L$ has Property (A) but also if L

has a finite number of atoms then the residue class ring $\mathcal{F}_{\mathcal{P}}L/\mathrm{Soc}(\mathcal{F}_{\mathcal{P}}L)$ has Property (A).

2 Preliminaries

For a general theory of frames we refer to [16, 24]. Here we collect a few facts that will be relevant for our discussion. A *frame* is a complete lattice L in which the infinite distributive law

$$x \wedge \bigvee S = \bigvee\{x \wedge s : s \in S\}$$

holds for all $x \in L$ and $S \subseteq L$. We denote the top element and the bottom element of L by $\top$ and $\bot$, respectively. The frame of all subsets of a set X is denoted by $P(X)$. A *frame homomorphism* (or *frame map*) is a map between frames which preserves finite meets, including the top element, and arbitrary joins, including the bottom element.

Let L be a lattice and $a \in L$. Then a is said to be **atom** if $a \neq \bot$ and there exists no element x with $\bot < x < a$. Also, an element a of a frame L is said to be *rather below* an element b, written $a \prec b$, in case there is an element s, called a *separating element*, such that $a \wedge s = \bot$ and $s \vee b = \top$. On the other hand, a is *completely below* b, written $a \prec\prec b$, if there are elements (c_q) indexed by the rational numbers $\mathbb{Q} \cap [0,1]$ such that $c_0 = a$, $c_1 = b$, and $c_p \prec c_q$ for $p < q$. A frame L is said to be *completely regular* if $a = \bigvee\{x \in L \mid x \prec\prec a\}$ for each $a \in L$.

We denote the *frame of reals* and the *ring of continuous of real-valued functions* on a completely regular frame L, by $\mathcal{L}(\mathbb{R})$ and $\mathcal{R}L$, respectively. Recall that $\mathcal{R}L$ is the collection of frame homomorphisms from $\mathcal{L}(\mathbb{R})$ into L (see [3, 4]). The *cozero map* (see [4, 5] for details) is the map $coz : \mathcal{R}L \to L$ given by

$$coz f = \bigvee\{f(p,0) \vee f(0,q) \,|\, p,q \in \mathbb{Q}\} = f((-\infty,0) \vee (0,+\infty)).$$

A *cozero element* of L in $\mathcal{R}L$ is an element of the form $coz(\alpha)$, for some $\alpha \in \mathcal{R}L$. A frame L is *completely regular* if and only if $Coz(\mathcal{R}L)$ generates L, where $Coz(\mathcal{R}L) = \{coz(\alpha) \,|\, \alpha \in \mathcal{R}L\}$.

A lattice-ordered ring (ℓ-ring) is a commutative ring A with the identity 1 whose underlying set is endowed with a lattice ordering such that for each $a, b, c \in A$, $(a \wedge b) + c = (a + c) \wedge (b + c)$, and $ab \geq 0$, whenever $a, b \geq 0$. An

f-ring is an ℓ-ring A which satisfies $(a \wedge b)c = (ac) \wedge (bc)$ for any $a, b \in A$ and $c \geq 0$ in A.

A real-valued function on a frame L is a frame homomorphism $f : P(\mathbb{R}) \to L$, where one assumes $(P(\mathbb{R}), \subseteq)$ to be a Boolean frame. The set of all real-valued functions on a frame L is denoted by $\mathcal{F}_{\mathcal{P}}L$. In [18] the authors showed that, the set $\mathcal{F}_{\mathcal{P}}L$ by operation $\diamond : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a sub-f-ring of $\mathcal{R}L$ in which for all $f, g \in \mathcal{F}_{\mathcal{P}}L$, $f \diamond g : P(\mathbb{R}) \to L$ defined by

$$(f \diamond g)(X) = \bigvee\{f(Y) \wedge g(Z) : Y \diamond Z \subset X\},$$

where $\diamond \in \{+, -, \wedge, \vee\}$ and $Y \diamond Z = \{y \diamond z : y \in Y,\ z \in Z\}$. For any frame L, the mapping $\mathcal{F}_{\mathcal{P}}L \to \mathcal{R}L$ taking any f to $f \circ j$ is an f-ring monomorphism, where $j : \mathcal{L}(\mathbb{R}) \to \mathfrak{O}(R)$ taking (p, q) to $]\!]p, q[\![:= \{x \in \mathbb{R} : p < x < q\}$ is an isomorphism (see [18, Theorem 6.1]).

The constant real-valued function on a frame L in $\mathcal{F}_{\mathcal{P}}L$ is

$$\mathbf{c}(X) = \begin{cases} \top & \text{if } c \in X \\ \bot & \text{if } c \notin X \end{cases},$$

for every $X \in P(\mathbb{R})$ and $c \in \mathbb{R}$. According to [27], for every $f \in \mathcal{F}_{\mathcal{P}}L$, $f(\{0\})$ is denoted by $z(f)$ and is called a *zero-element*. Any element in L which is a zero-element of some frame map in $\mathcal{F}_{\mathcal{P}}L$ is called a zero-element of L. Thus, z is a mapping from the ring $\mathcal{F}_{\mathcal{P}}L$ onto the set of all zero-elements in L. Also a cozero-element of L in $\mathcal{F}_{\mathcal{P}}L$ is defined by $coz(f) := f(\mathbb{R} \setminus \{0\})$ for some $f \in \mathcal{F}_{\mathcal{P}}L$. It is clear that $z(f) = (coz(f))'$. Now we recall some properties of $\mathcal{F}_{\mathcal{P}}L$ which will be used in the sequel.

Theorem 2.1. [27] *For every $f, g \in \mathcal{F}_{\mathcal{P}}L$, we have*

(1) *for every $n \in \mathbb{N}$, $z(f) = z(-f) = z(|f|) = z(f^n)$,*
(2) *$z(fg) = z(f) \vee z(g)$,*
(3) *$z(f + g) \geq z(f) \wedge z(g)$,*
(4) *$z(f + g) = z(f) \wedge z(g)$, while $f, g \geq \mathbf{0}$,*
(5) *$z(f) = \top$ if and only if $f = \mathbf{0}$,*
(6) *$z(f) = \bot$ if and only if f is a unit element of $\mathcal{F}_{\mathcal{P}}L$.*

Proof. We prove the last assertion. Suppose that f is a unit of $\mathcal{F}_{\mathcal{P}}L$. Then there exists $g \in \mathcal{F}_{\mathcal{P}}L$ such that $fg = \mathbf{1}$. So by part (2), $\bot = z(1) = z(fg) = z(f) \vee z(g)$, and hence $z(f) = \bot$.

Conversely, assume that $f \in \mathcal{F}_{\mathcal{P}}L$ and $z(f) = \bot$. Define

$$g(X) := \bigvee\{f\{\frac{1}{x}\} \mid x \in X - \{0\}\}\,.$$

We show that g belongs to $\mathcal{F}_{\mathcal{P}}L$ which is the multiplicative inverse of f in $\mathcal{F}_{\mathcal{P}}L$. The proof consists of four steps to check:
Step 1. The first step is verifying that $g(\mathbb{R}) = \top$. Since $f\{0\} = \bot$, we have

$$\begin{aligned} g(\mathbb{R}) &= \bigvee\{f\{\tfrac{1}{x}\} \mid x \in \mathbb{R} - \{0\}\} \\ &= \bot \vee \bigvee\{f\{\tfrac{1}{x}\} \mid x \in \mathbb{R} - \{0\}\} \\ &= f\{0\} \vee \bigvee\{f\{\tfrac{1}{x}\} \mid x \in \mathbb{R} - \{0\}\} \\ &= f(\mathbb{R}) \\ &= \top. \end{aligned}$$

Step 2. Let $\{X_i\}_{i\in I} \subseteq P(\mathbb{R})$. If for all i, $X_i = \emptyset$ or $\{0\}$, then obviously,

$$g(\bigcup_{i\in I} X_i) = \bot = \bigvee_{i\in I} g(X_i),$$

or else there is an i which $X_i \neq \emptyset, \{0\}$, then

$$\begin{aligned} g(\textstyle\bigcup_{i\in I} X_i) &= \bigvee\{f\{\tfrac{1}{x}\} \mid x \in (\textstyle\bigcup_{i\in I} X_i) - \{0\}\} \\ &= \bigvee\{f\{\tfrac{1}{x}\} \mid x \in \textstyle\bigcup_{i\in I}(X_i - \{0\})\} \\ &= \textstyle\bigvee_{i\in I} \bigvee\{f\{\tfrac{1}{x}\} \mid x \in X_i - \{0\}\} \\ &= \textstyle\bigvee_{i\in I} g(X_i)\,. \end{aligned}$$

Step 3. Let $X, Y \in P(\mathbb{R})$. If $X, Y \in \{\emptyset, \{0\}\}$, then obviously,

$$g(X \cap Y) = \bot = g(X) \wedge g(Y),$$

or else we have

$$\begin{aligned} g(X \cap Y) &= \bigvee\{f\{\tfrac{1}{x}\} \mid x \in (X \cap Y) - \{0\}\} \\ &= \bigvee\{f\{\tfrac{1}{x}\} \mid x \in (X - \{0\}) \cap (Y - \{0\})\} \\ &= \bigvee\{f\{\tfrac{1}{x}\} \wedge f\{\tfrac{1}{y}\} \mid x \in X - \{0\}, y \in Y - \{0\}\} \\ &= \bigvee\{f\{\tfrac{1}{x}\} \mid x \in X - \{0\}\} \wedge \bigvee\{f\{\tfrac{1}{y}\} \mid y \in Y - \{0\}\} \\ &= g(X) \wedge g(Y)\,. \end{aligned}$$

Step 4. In the last step, we show that $fg = \mathbf{1}$. We have

$$\begin{aligned}(fg)(\{1\}) &= \bigvee\{f(\{x\}) \wedge g(\{y\}) \mid xy = 1\} \\ &= \bigvee\{f(\{x\}) \wedge g(\{\tfrac{1}{x}\}) \mid 0 \neq x \in \mathbb{R}\} \\ &= \bigvee\{f(\{x\}) \wedge f(\{x\}) \mid 0 \neq x \in \mathbb{R}\} \\ &= \bigvee\{f(\{x\}) \mid 0 \neq x \in \mathbb{R}\} \\ &= f(\{0\}) \bigvee\{f(\{x\}) \mid 0 \neq x \in \mathbb{R}\} \\ &= f(\mathbb{R}) \\ &= \top\end{aligned}$$

and

$$\begin{aligned}(fg)(\{0\}) &= \bigvee\{f(\{x\}) \wedge g(\{y\}) \mid xy = 0\} \\ &= \bigvee\{f(\{x\}) \wedge g(\{y\}) \mid x = 0\} \vee \bigvee\{f(\{x\}) \wedge g(\{y\}) \mid y = 0\} \\ &= \bigvee\{\bot \wedge g(\{y\})\} \vee \bigvee\{f(\{x\}) \wedge \bot\} \\ &= \bot.\end{aligned}$$

Also, if $r \neq 0, 1$, then

$$\begin{aligned}(fg)(\{r\}) &= \bigvee\{f(\{x\}) \wedge g(\{y\}) \mid xy = r\} \\ &= \bigvee\{f(\{x\}) \wedge g(\{\tfrac{r}{x}\}) \mid 0 \neq x \in \mathbb{R}\} \\ &= \bigvee\{f(\{x\}) \wedge f(\{\tfrac{x}{r}\}) \mid x \neq 0\} \\ &= \bigvee\{f(\emptyset) \mid x \neq 0\} \\ &= \bot\end{aligned}$$

and thus $fg = \mathbf{1}$. The proof is now complete. □

3 On minimal ideals of $\mathcal{F}_{\mathcal{P}}L$

We recall from [20, p. 63] that a minimal ideal of a *reduced* ring (a ring without any nonzero nilpotent element) is generated by an idempotent. Furthermore, if R is a reduced ring and $e^2 = e \in R$, then eR is a minimal ideal if and only if eR is a field with the multiplicative identity e. In this section, we study minimal ideals of $\mathcal{F}_{\mathcal{P}}L$ and we show that if I is a minimal ideal of $\mathcal{F}_{\mathcal{P}}L$ and $a = \bigvee_{f\in I} coz(f)$ then I is generated by f_a, where f_a is introduced in the following proposition.

Proposition 3.1. *Let a be a complemented element of the frame L. Then $f_a : P(\mathbb{R}) \to L$, defined by*

$$f_a(X) = \begin{cases} \top & \text{if } 0, 1 \in X \\ a' & \text{if } 0 \in X \text{ and } 1 \notin X \\ a & \text{if } 0 \notin X \text{ and } 1 \in X \\ \bot & \text{if } 0 \notin X \text{ and } 1 \notin X, \end{cases}$$

is a real-valued function on L.

Proof. Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of subsets of $\mathbb{R}$. Then

$$\begin{aligned} f_a(\bigcup_{\lambda\in\Lambda} X_\lambda) &= \begin{cases} \top & \text{if } \exists\lambda_1, \lambda_2 \in \Lambda \text{ such that } 0 \in X_{\lambda_1} \text{ and } 1 \in X_{\lambda_2}, \\ a' & \text{if } 0 \in X_\lambda \text{ and } 1 \notin X_\lambda, \text{ for every } \lambda \in \Lambda, \\ a & \text{if } 0 \notin X_\lambda \text{ and } 1 \in X_\lambda, \text{ for every } \lambda \in \Lambda, \\ \bot & \text{if } 0 \notin X_\lambda \text{ and } 1 \notin X_\lambda, \text{ for every } \lambda \in \Lambda, \end{cases} \\ &= \bigvee_{\lambda\in\Lambda} f_a(X_\lambda). \end{aligned}$$

A straightforward calculation shows that $f_a(A \cap B) = f_a(A) \wedge f_a(B)$, for every $A, B \in P(\mathbb{R})$. Since $f_a(\mathbb{R}) = \top$ and $f_a(\emptyset) = \bot$, we conclude that f_a is a real-valued function on L. □

From now on, unless specified otherwise, f_a denotes the real-valued function from the power set of $\mathbb{R}$ into L, defined in Proposition 3.1.

Proposition 3.2. *Let a be a complemented element of L. Then $f_a^2 = f_a$.*

Proof. Let x be a nonzero element of $\mathbb{R}$. Then

$$\begin{aligned} f_a^2(\{x\}) &= \bigvee_{0\neq y\in\mathbb{R}} f_a(\{y\}) \wedge f_a(\{\tfrac{x}{y}\}) \\ &= \bigvee_{0\neq y\in\mathbb{R}} f_a(\{y\} \cap \{\tfrac{x}{y}\}) \\ &= \begin{cases} f_a(\{1\}) & \text{if } x = 1, \\ \bot & \text{if } x \neq 1 \end{cases} \\ &= f_a(\{x\}). \end{aligned}$$

Since $z(f_a^2) = z(f_a)$, we conclude that $f_a^2(\{0\}) = f_a(\{0\})$. Hence $f_a^2 = f_a$. □

Proposition 3.3. *Let a be a complemented element of L. Then for every $f \in \mathcal{F}_{\mathcal{P}}L$ and $X \in P(\mathbb{R})$,*

$$ff_a(X) = \begin{cases} a' \vee f(X) & \text{if } 0 \in X, \\ a \wedge f(X) & \text{if } 0 \notin X. \end{cases}$$

Proof. By Theorem 2.1, we have

$$ff_a(\{0\}) = z(ff_a) = z(f) \vee z(f_a) = f(\{0\}) \vee a'.$$

Now, let x be a nonzero element of $\mathbb{R}$. Then

$$\begin{array}{rcl} ff_a(\{x\}) & = & \bigvee_{0 \neq y \in \mathbb{R}} f(\{y\}) \wedge f_a(\{\frac{x}{y}\}) \\ & = & f(\{x\}) \wedge f_a(\{1\}) \\ & = & f(\{x\}) \wedge a. \end{array}$$

We consider the following two cases:

Case 1: Let $X \in P(\mathbb{R})$ with $0 \in X$. Then

$$\begin{array}{rcl} ff_a(X) & = & ff_a(\{0\}) \vee ff_a(X \setminus \{0\}) \\ & = & (f(\{0\}) \vee a') \vee \bigvee_{x \in X \setminus \{0\}} ff_a(\{x\}) \\ & = & (f(\{0\}) \vee a') \vee \bigvee_{x \in X \setminus \{0\}} f(\{x\}) \wedge a \\ & = & (f(\{0\}) \vee a') \vee (a \wedge \bigvee_{x \in X \setminus \{0\}} f(\{x\})) \\ & = & (f(\{0\}) \vee a') \vee (a \wedge f(X \setminus \{0\})) \\ & = & (f(\{0\}) \vee a' \vee a) \wedge (f(\{0\}) \vee a' \vee f(X \setminus \{0\})) \\ & = & \top \wedge (a' \vee f(X)) \\ & = & a' \vee f(X). \end{array}$$

Case 2: Let $X \in P(\mathbb{R})$ with $0 \notin X$. Then

$$\begin{array}{rcl} ff_a(X) & = & \bigvee_{x \in X} ff_a(\{x\}) \\ & = & \bigvee_{x \in X} f(\{x\}) \wedge a \\ & = & a \wedge \bigvee_{x \in X} f(\{x\}) \\ & = & a \wedge f(X). \end{array}$$

Hence

$$ff_a(X) = \begin{cases} a' \vee f(X) & \text{if } 0 \in X, \\ a \wedge f(X) & \text{if } 0 \notin X, \end{cases}$$

for every $X \subseteq \mathbb{R}$. □

For an element a of a frame L, let $\mathfrak{R}_{\mathcal{F}_{\mathcal{P}}}(a) := \{f \in \mathcal{F}_{\mathcal{P}}L \,|\, coz(f) \leq a\}$. Clearly $\mathfrak{R}_{\mathcal{F}_{\mathcal{P}}}(a)$ is an ideal of $\mathcal{F}_{\mathcal{P}}L$.

Proposition 3.4. *Let a be a complemented element of a frame L. Then $\mathfrak{R}_{\mathcal{F}_{\mathcal{P}}}(a)$ is a principal ideal generated by f_a.*

Proof. Suppose that $0 \neq f \in \mathfrak{R}_{\mathcal{F}_{\mathcal{P}}}(a)$. Then $coz(f) \leq a$ and $z(f) \geq a'$. Let $X \in P(\mathbb{R})$ with $0 \in X$. Then $a' \leq f(\{0\}) \leq f(X)$ and, using Proposition 3.3, we have

$$f(X) = a' \vee f(X) = ff_a(X).$$

Now, we assume that $X \in P(\mathbb{R})$ with $0 \notin X$. Since

$$f(X) \subseteq f(\mathbb{R} \setminus \{0\}) = coz(f) \leq a,$$

we conclude from Proposition 3.3 that $f(X) = F(X) \wedge a = ff_a(X)$. Hence $\mathfrak{R}_{\mathcal{F}_{\mathcal{P}}}(a) \subseteq < f_a >$. Evidently, $< f_a > \subseteq \mathfrak{R}_{\mathcal{F}_{\mathcal{P}}}(a)$, since $coz(f_a) = a$. □

Remark 3.5. We have the following conclusions:

(1) $\mathfrak{R}_{\mathcal{F}_{\mathcal{P}}}(\top) = \mathcal{F}_{\mathcal{P}}L$ and $\mathfrak{R}_{\mathcal{F}_{\mathcal{P}}}(\bot) = (0)$.

(2) For each pair of complemented elements $a, b \in L$, $f_a f_b = f_{a \wedge b}$.

(3) $f_a + f_{a'} = 1$.

Lemma 3.6. *If a is an atom element of a frame L and $0 \neq g \in \mathfrak{R}_{\mathcal{F}_{\mathcal{P}}}(a)$, then $h : P(\mathbb{R}) \to L$ defined by*

$$h(X) = \begin{cases} a' \vee \bigvee_{0 \neq x \in X} g(\{\frac{1}{x}\}) & \textit{if } 0 \in X, \\ a \wedge \bigvee_{x \in X} g(\{\frac{1}{x}\}) & \textit{if } 0 \notin X, \end{cases}$$

is a real-valued function on L.

Proof. Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of subsets of $\mathbb{R}$. We put $\Lambda_0 := \{\lambda \in \Lambda \,|\, 0 \in X_\lambda\}$ and $\Lambda_1 := \{\lambda \in \Lambda \,|\, 0 \notin X_\lambda\}$. Then

$$\begin{aligned} \bigvee_{\lambda \in \Lambda} h(X_\lambda) &= \bigvee_{\lambda \in \Lambda_0} h(X_\lambda) \vee \bigvee_{\lambda \in \Lambda_1} h(X_\lambda) \\ &= \bigvee_{\lambda \in \Lambda_0} (a' \vee \bigvee_{0 \neq x \in X_\lambda} g(\{\tfrac{1}{x}\})) \vee \bigvee_{\lambda \in \Lambda_1} (a \wedge \bigvee_{x \in X_\lambda} g(\{\tfrac{1}{x}\})) \\ &= (a' \vee \bigvee_{\lambda \in \Lambda_0} \bigvee_{0 \neq x \in X_\lambda} g\{\tfrac{1}{x}\}) \vee (a \wedge \bigvee_{\lambda \in \Lambda_1} \bigvee_{x \in X_\lambda} g\{\tfrac{1}{x}\}) \\ &= (a' \vee \bigvee_{0 \neq x \in \bigcup_{\lambda \in \Lambda_0} X_\lambda} g\{\tfrac{1}{x}\}) \vee (a \wedge \bigvee_{x \in \bigcup_{\lambda \in \Lambda_1} X_\lambda} g\{\tfrac{1}{x}\}) \\ &= (a' \vee \bigvee_{0 \neq x \in \bigcup_{\lambda \in \Lambda_0} X_\lambda} g\{\tfrac{1}{x}\} \vee a) \wedge (a' \vee \bigvee_{0 \neq x \in \bigcup_{\lambda \in \Lambda} X_\lambda} g\{\tfrac{1}{x}\}) \\ &= h(\bigcup_{\lambda \in \Lambda} X_\lambda). \end{aligned}$$

Assume that $A, B \in P(\mathbb{R})$. If $0 \notin A$ and $0 \notin B$, then

$$\begin{aligned} h(A) \wedge h(B) &= (a \wedge \bigvee_{x \in A} g(\{\tfrac{1}{x}\})) \wedge (a \wedge \bigvee_{y \in B} g(\{\tfrac{1}{y}\})) \\ &= a \wedge \bigvee_{x \in A, y \in B} (g(\{\tfrac{1}{x}\}) \wedge g(\{\tfrac{1}{y}\})) \\ &= a \wedge \bigvee_{x \in A, y \in B} (g(\{\tfrac{1}{x}\} \cap \{\tfrac{1}{y}\})) \\ &= a \wedge \bigvee_{x \in A \cap B} g(\{\tfrac{1}{x}\}) \\ &= h(A \cap B). \end{aligned}$$

If $0 \in A$ and $0 \notin B$, then

$$\begin{aligned} h(A) \wedge h(B) &= (a' \vee \bigvee_{0 \neq x \in A} g(\{\tfrac{1}{x}\})) \wedge (a \wedge \bigvee_{y \in B} g(\{\tfrac{1}{y}\})) \\ &= (a' \wedge a \wedge \bigvee_{y \in B} g\{\tfrac{1}{y}\}) \vee (a \wedge \bigvee_{0 \neq x \in A, y \in B} g(\{\tfrac{1}{x}\} \cap \{\tfrac{1}{y}\})) \\ &= a \wedge \bigvee_{x \in A \cap B} g(\{\tfrac{1}{x}\}) \\ &= h(A \cap B). \end{aligned}$$

If $0 \in A$ and $0 \in B$, then

$$\begin{aligned} h(A) \wedge h(B) &= (a' \vee \bigvee_{0 \neq x \in A} g(\{\tfrac{1}{x}\})) \wedge (a' \vee \bigvee_{0 \neq y \in B} g(\{\tfrac{1}{y}\})) \\ &= a' \vee (\bigvee_{0 \neq x \in A} g(\{\tfrac{1}{x}\}) \wedge \bigvee_{0 \neq y \in B} g(\{\tfrac{1}{y}\})) \\ &= a' \vee \bigvee_{0 \neq x \in A, 0 \neq y \in B} g(\{\tfrac{1}{x}\} \cap \{\tfrac{1}{y}\}) \\ &= a' \vee \bigvee_{0 \neq x \in A \cap B} g(\{\tfrac{1}{x}\}) \\ &= h(A \cap B). \end{aligned}$$

Therefore, $h(A \cap B) = h(A) \wedge h(B)$, for every $A, B \in P(\mathbb{R})$. Also, since a is an atom element of L and $\perp < coz(g) \leq a$, we conclude that $coz(g) = a$. Finaly, from

$$h(\mathbb{R}) = a' \vee \bigvee_{0 \neq x \in A} g(\{\frac{1}{x}\}) = a' \vee coz(g) = a' \vee a = \top$$

and

$$h(\emptyset) = a \wedge \bigvee_{x \in \emptyset} g(\{\frac{1}{x}\}) = a \wedge \perp = \perp,$$

we infer that h is a real-valued function on L. □

Lemma 3.7. *Let a be an atom element of a frame L and $0 \neq g \in \mathfrak{R}_{\mathcal{F}_{\mathcal{P}}}(a)$. If $h \in \mathfrak{R}_{\mathcal{F}_{\mathcal{P}}}(a)$ is the same function of Lemma 3.6, then $hg = f_a$.*

Proof. Let X be a subset of $\mathbb{R}$. If $\{0,1\} \subseteq X$, then

$$\begin{aligned}
hg(X) &= hg(\{0\}) \vee hg(\{1\}) \vee hg(X \setminus \{0,1\}) \\
&= (h\{0\} \vee g\{0\}) \vee (\textstyle\bigvee_{0 \neq x \in \mathbb{R}}(h\{x\} \wedge g\{\frac{1}{x}\})) \vee hg(X \setminus \{0,1\}) \\
&= (a' \vee a') \vee (\textstyle\bigvee_{0 \neq x \in \mathbb{R}}(a \wedge g\{\frac{1}{x}\} \wedge g\{\frac{1}{x}\})) \vee hg(X \setminus \{0,1\}) \\
&= a' \vee (a \wedge \textstyle\bigvee_{0 \neq x \in \mathbb{R}} g\{\frac{1}{x}\}) \vee hg(X \setminus \{0,1\}) \\
&= a' \vee (a \wedge coz(g)) \vee hg(X \setminus \{0,1\}) \\
&= a' \vee a \vee hg(X \setminus \{0,1\}) \\
&= \top \\
&= f_a(X).
\end{aligned}$$

If $0 \in X$ and $1 \notin X$, then

$$\begin{aligned}
hg(X) &= hg(\{0\}) \vee hg(X \setminus \{0\}) \\
&= (h(\{0\}) \vee g(\{0\})) \vee (\textstyle\bigvee_{0 \neq xy \in X}(h(\{x\}) \wedge g(\{y\}))) \\
&= a' \vee (\textstyle\bigvee_{0 \neq x \in \mathbb{R}}(a \wedge g(\{\frac{1}{x}\}) \wedge g(\{y\}))) \\
&= a' \vee (\textstyle\bigvee_{0 \neq x \in \mathbb{R}}(a \wedge g(\{\frac{1}{x}\} \cap \{y\}))) \\
&= a' \vee (a \wedge \bot) \\
&= a' \\
&= f_a(X).
\end{aligned}$$

If $0 \notin X$ and $1 \in X$, then

$$\begin{aligned}
hg(X) &= hg(\{1\}) \vee hg(X \setminus \{1\}) \\
&= (\textstyle\bigvee_{0 \neq x \in \mathbb{R}}(h(\{x\}) \wedge g(\{\frac{1}{x}\}))) \vee (\textstyle\bigvee_{1 \neq xy \in X}(h(\{x\}) \wedge g(\{y\}))) \\
&= (\textstyle\bigvee_{0 \neq x \in \mathbb{R}}(a \wedge g\{\frac{1}{x}\} \wedge g\{\frac{1}{x}\})) \vee (\textstyle\bigvee_{1 \neq xy \in X}(a \wedge g\{\frac{1}{x}\} \wedge g\{y\})) \\
&= (a \wedge \textstyle\bigvee_{0 \neq x \in \mathbb{R}}(g(\{\frac{1}{x}\}))) \vee (\textstyle\bigvee_{1 \neq xy \in X}(a \wedge g(\{\frac{1}{x}\} \cap \{y\}))) \\
&= (a \wedge coz(g)) \vee (\textstyle\bigvee_{1 \neq xy \in X}(a \wedge \bot)) \\
&= a \\
&= f_a(X).
\end{aligned}$$

If $0 \notin X$ and $1 \notin X$, then

$$\begin{aligned} hg(X) &= \bigvee_{xy \in X}(h(\{x\}) \wedge g(\{y\})) \\ &= \bigvee_{xy \in X}(a \wedge g(\{\tfrac{1}{x}\}) \wedge g(\{y\})) \\ &= \bigvee_{xy \in X}(a \wedge g(\{\tfrac{1}{x}\} \cap \{y\})) \\ &= \bigvee_{xy \in X}(a \wedge \bot) \\ &= \bot \\ &= f_a(X). \end{aligned}$$

Hence $hg = f_a$. □

Proposition 3.8. *Let a be an atom of a frame L. Then $\mathfrak{R}_{\mathcal{F}_\mathcal{P}}(a)$ is a minimal ideal of $\mathcal{F}_\mathcal{P}L$.*

Proof. By Lemmas 3.6 and 3.7, $\mathfrak{R}_{\mathcal{F}_\mathcal{P}}(a)$ is a field with the multiplicative identity f_a. Hence, by Propositions 3.2 and 3.4, $\mathfrak{R}_{\mathcal{F}_\mathcal{P}}(a)$ is a minimal ideal of $\mathcal{F}_\mathcal{P}L$. □

Let R be a commutative ring with unit. We let $\mathscr{M}$ denote its maximal ideal space of R and put $\mathscr{M}(a) = \{M \in \mathscr{M} \mid a \in M\}$ for all $a \in R$. An ideal I of R is called a *z-ideal* if $\mathscr{M}(a) = \mathscr{M}(b)$ and $a \in I$, then $b \in I$. Equivalently, since $\mathscr{M}(a) \subseteq \mathscr{M}(b)$ if and only if $\mathscr{M}(ab) = \mathscr{M}(b)$, hence I is a z-ideal if and only if $\mathscr{M}(a) \subseteq \mathscr{M}(b)$ and $a \in I$ implies $b \in I$ (see [23]). It is well known that if R has minimal nonzero ideals, then they are z-ideals (see [23]).

Lemma 3.9. *Let I be a minimal ideal of $\mathcal{F}_\mathcal{P}L$. Then $I = \mathfrak{R}_{\mathcal{F}_\mathcal{P}}(a)$, for some complemented element a of L.*

Proof. Clearly every minimal ideal of $\mathcal{F}_\mathcal{P}L$ is generated by an idempotent. Hence there exists an idempotent $e \in I$ such that $I = e\mathcal{F}_\mathcal{P}L$. Now consider $a = coz(e)$. Since I is a z-ideal and $z(f_a) = a' = z(e)$, we conclude that $f_a \in I$. Proposition 3.4 insures that $\mathfrak{R}_{\mathcal{F}_\mathcal{P}}(a) \subseteq I$ and the minimality of I implies that $I = \mathfrak{R}_{\mathcal{F}_\mathcal{P}}(a)$. □

Definition 3.10. A frame L is called an $\mathcal{F}_\mathcal{P}$-completely regular frame provided there exists $\mathcal{A} \subseteq \mathcal{F}_\mathcal{P}$ such that $a = \bigvee_{f \in \mathcal{A}} coz(f)$, for every $a \in L$.

If L is an $\mathcal{F}_{\mathcal{P}}$-completely regular frame and $a \in L$, then there exists $\mathcal{A} \subseteq \mathcal{F}_{\mathcal{P}}$ such that $a = \bigvee_{f\in\mathcal{A}} coz(f) = \bigvee_{f\in\mathcal{A}} coz(f \circ j)$, which shows that L is a completely regular frame.

Proposition 3.11. *Let L be an $\mathcal{F}_{\mathcal{P}}$-completely regular frame. Then for an ideal I of $\mathcal{F}_{\mathcal{P}}L$, the following statements are equivalent.*

(1) *The ideal I is a minimal ideal of $\mathcal{F}_{\mathcal{P}}L$.*

(2) *$I = \mathfrak{R}_{\mathcal{F}_{\mathcal{P}}}(a)$, for some atom a of L.*

Proof. (1)⇒(2) There exists a complemented element a of L such that $I = \mathfrak{R}_{\mathcal{F}_{\mathcal{P}}}(a)$, by Lemma 3.9. Suppose that there exists $s \in L$ such that $\perp < s \leq a$. Since L is an $\mathcal{F}_{\mathcal{P}}$-completely regular frame, we conclude that there exists $g \in \mathcal{F}_{\mathcal{P}}L$ such that $\perp < coz(g) \leq s \leq a$, which shows that $0 \neq g \in I$, and so $I = < g >$, because I is a minimal ideal of $\mathcal{F}_{\mathcal{P}}L$. Hence there exists $h \in \mathcal{F}_{\mathcal{P}}L$ such that $f_a = hg$, which implies that $a = coz(f_a) \leq coz(g)$ and so $coz(g) = s = a$. Therefore, a is an atom and $I = \mathfrak{R}_{\mathcal{F}_{\mathcal{P}}}(a)$.

(2)⇒(1) By Proposition 3.8, it is evident. □

4 On Property (A) of $\mathcal{F}_{\mathcal{P}}L$

Recall that a commutative ring R has Property (A) if every finitely generated ideal of R consisting entirely of zero-divisors has a nonzero annihilator. In [2] the authors showed that $C(X)$ has Property (A). In this section, we show that (i) $\mathcal{F}_{\mathcal{P}}L$ has Property (A), (ii) if L has a finite number of atoms, then the residue class ring $\mathcal{F}_{\mathcal{P}}L/\mathrm{Soc}(\mathcal{F}_{\mathcal{P}}L)$ has Property (A).

Proposition 4.1. *Let L be a frame. Then f-ring $\mathcal{F}_{\mathcal{P}}L$ has Property (A).*

Proof. Let $I = \sum_{i=1}^n f_i\mathcal{F}_{\mathcal{P}}L \subseteq Z(\mathcal{F}_{\mathcal{P}}L)$. Since $f = \sum_{i=1}^n f_i^2 \in I$, we conclude that there exists $0 < g \in \mathcal{F}_{\mathcal{P}}L$ such that $fg = 0$, which shows that $coz(fg) = \perp$. Let $h = \sum_{i=1}^n f_ih_i \in I$. Then

$$\begin{aligned} coz(hg) &= coz(\textstyle\sum_{i=1}^n f_ih_ig) \\ &\leq \textstyle\bigvee_{i=1}^n coz(|f_i||h_i|g) \\ &\leq \textstyle\bigvee_{i=1}^n coz(f_i^2 g) \\ &= coz(\textstyle\sum_{i=1}^n f_i^2 g) \\ &= coz(fg) \\ &= \perp. \end{aligned}$$

Hence $hg = 0$, which implies that $g \in Ann(I)$. Therefore, $\mathcal{F}_{\mathcal{P}}L$ has Property (A). □

Lemma 4.2. *Let $a_1, \ldots, a_n$ be disjoint atomic elements of a frame L. Then for every $f \in \mathcal{F}_{\mathcal{P}}L$ and $X \in P(\mathbb{R})$,*

$$\sum_{i=1}^{n} ff_{a_i}(X) = \begin{cases} (\bigwedge_{i=1}^{n} a'_i) \vee f(X) & \text{if } \ 0 \in X, \\ (\bigvee_{i=1}^{n} a_i) \wedge f(X) & \text{if } \ 0 \notin X. \end{cases}$$

Proof. We prove it by induction on the number of complemented elements. For $n = 1$, it follows from Proposition 3.3. For $n > 1$ let $g = \sum_{i=1}^{n-1} ff_{a_i}$. Then

$$g(\{0\}) \wedge ff_{a_n}(\{0\}) = [(\bigwedge_{i=1}^{n-1} a'_i) \vee f(\{0\})] \wedge [a'_n \vee f(\{0\})] = (\bigwedge_{i=1}^{n} a'_i) \vee f(\{0\}),$$

$$\begin{aligned} g(\{x\}) \wedge ff_{a_n}(\{0\}) &= [(\bigvee_{i=1}^{n-1} a_i) \wedge f(\{x\})] \wedge [a'_n \vee f(\{0\})] \\ &= [(\bigvee_{i=1}^{n-1} a_i) \wedge f\{x\} \wedge a'_n] \vee [(\bigvee_{i=1}^{n-1} a_i) \wedge f\{x\} \wedge f\{0\}] \\ &= (\bigvee_{i=1}^{n-1} a_i) \wedge f(\{x\}), \\ g(\{0\}) \wedge ff_{a_n}(\{x\}) &= [(\bigwedge_{i=1}^{n-1} a'_i) \vee f(\{0\})] \wedge [a_n \wedge f(\{x\})] \\ &= [(\bigwedge_{i=1}^{n-1} a'_i) \wedge a_n \wedge f(\{x\})] \vee [f(\{0\}) \wedge a_n \wedge f(\{x\})] \\ &= a_n \wedge f(\{x\}), \end{aligned}$$

for every $0 \neq x \in \mathbb{R}$. Also if $x, y \in \mathbb{R}$ with $y \neq 0 \neq x$, then

$$g(\{x\}) \wedge ff_{a_n}(\{y\}) = [(\bigvee_{i=1}^{n-1} a_i) \wedge f(\{x\})] \wedge [a_n \wedge f(\{y\})] = \bot.$$

Hence

$$(g+ff_{a_n})(\{0\}) = \bigvee_{x\in\mathbb{R}} g\{x\} \wedge ff_{a_n}(\{-x\}) = g\{0\} \wedge ff_{a_n}(\{0\}) = (\bigwedge_{i=1}^{n} a'_i) \vee f\{0\}$$

and

$$\begin{aligned} (g + ff_{a_n})(\{x\}) &= \bigvee_{y\in\mathbb{R}} g(\{y\}) \wedge ff_{a_n}(\{x - y\}) \\ &= [g(\{0\}) \wedge ff_{a_n}(\{x\})] \vee [g(\{x\}) \wedge ff_{a_n}(\{0\})] \\ &= [a_n \wedge f(\{x\})] \vee [(\bigvee_{i=1}^{n-1} a_i) \wedge f(\{x\})] \\ &= (\bigvee_{i=1}^{n} a_i) \wedge f(\{x\}). \end{aligned}$$

Thus if $0 \in X \subseteq \mathbb{R}$, then

$$\begin{aligned}(\textstyle\sum_{i=1}^{n} ff_{a_i})(X) &= (g + ff_{a_n})(\{0\}) \vee \textstyle\bigvee_{0 \neq x \in X}(g + ff_{a_n})(\{x\}) \\ &= [(\textstyle\bigwedge_{i=1}^{n} a_i') \vee f(\{0\})] \vee \textstyle\bigvee_{0 \neq x \in X}[(\textstyle\bigvee_{i=1}^{n} a_i) \wedge f(\{x\})] \\ &= [(\textstyle\bigwedge_{i=1}^{n} a_i') \vee f(\{0\})] \vee [(\textstyle\bigvee_{i=1}^{n} a_i) \wedge \textstyle\bigvee_{0 \neq x \in X} f(\{x\})] \\ &= (\textstyle\bigwedge_{i=1}^{n} a_i') \vee f(X),\end{aligned}$$

and if $0 \notin X \subseteq \mathbb{R}$, then

$$(\sum_{i=1}^{n} ff_{a_i})(X) = \bigvee_{x \in X} (g+ff_{a_n})(\{x\}) = \bigvee_{x \in X} [(\bigvee_{i=1}^{n} a_i) \wedge f(\{x\})] = (\bigvee_{i=1}^{n} a_i) \wedge f(X).$$

This completes the induction. □

Corollary 4.3. *Let $a_1, \ldots, a_n$ be disjoint atomic elements of a frame L. Then for every $f \in \mathcal{F}_{\mathcal{P}}L$ and $X \in P(\mathbb{R})$,*

$$\sum_{i=1}^{n} f_{a_i} = f_b,$$

where $b = \bigvee_{i=1}^{n} a_i$.

Proof. Consider $b = \bigvee_{i=1}^{n} a_i$. Hence

$$(\sum_{i=1}^{n} f_{a_i})(A) = \begin{cases} 1(A) \vee b' & \text{if } 0 \in A, \\ 1(A) \wedge b & \text{if } 0 \notin A, \end{cases} = f_b(A),$$

for all $A \subseteq \mathbb{R}$.

□

Proposition 4.4. *Let L be an $\mathcal{F}_{\mathcal{P}}$-completely regular frame. Then the socle of $\mathcal{F}_{\mathcal{P}}L$ consists of those f for which $coz(f)$ is a join of finitely many atoms.*

Proof. If $\mathrm{Soc}(\mathcal{F}_{\mathcal{P}}L) = 0$, then there is nothing to prove. Now suppose that it is nonzero. If $f \in \mathrm{Soc}(\mathcal{F}_{\mathcal{P}}L)$, then there exist atoms $a_1, \ldots, a_k \in L$ and $f_1, \ldots, f_i \in \mathcal{F}_{\mathcal{P}}L$ such that

$$f = f_1 f_{a_1} + f_2 f_{a_2} + \cdots + f_k f_{a_k},$$

by Propositions 3.4 and 3.11, which implies that $coz(f) \leq \bigvee_{i=1}^{k} a_i$. Consequently, $coz(f) = \bigvee_{i=1}^{k} a_i \wedge coz(f)$. Since each a_i is an atom, hence $a_i \wedge coz(f) = 0$ or a_i, which implies that $coz(f)$ is a join of finitely many atoms.

Conversely, suppose that $coz(f) = \bigvee_{i=1}^{k} a_i$, where each a_i is an atom. By Propositions 3.4 and 3.11, $R_{\mathcal{F}_{\mathcal{P}}}(a_i)$ is a minimal ideal generated by f_{a_i}. If $0 \notin X \subseteq \mathbb{R}$, then, by Lemma 4.2, we have

$$(\sum_{i=1}^{n} f f_{a_i})(X) = (\bigvee_{i=1}^{n} a_i) \wedge f(X) = f(X),$$

since $f(X) \leq coz(f)$. If $0 \in X \subseteq \mathbb{R}$, then

$$(\sum_{i=1}^{n} f f_{a_i})(X) = ((\sum_{i=1}^{n} f f_{a_i})(\mathbb{R} \setminus X))' = (f(\mathbb{R} \setminus X))' = f(X).$$

Hence $f = \sum_{i=1}^{n} f f_{a_i} \in \sum_{i=1}^{n} \mathfrak{R}_{\mathcal{F}_{\mathcal{P}}}(a_i) \subseteq \mathrm{Soc}(\mathcal{F}_{\mathcal{P}} L)$. □

An element f of $\mathcal{F}_{\mathcal{P}} L$ is said to be *bounded* if $|f| \leq \mathbf{n}$, for some $n \in \mathbb{N}$. The set of all bounded real-valued functions on a frame L is denoted by $\mathcal{F}_{\mathcal{P}}^{*} L$.

Proposition 4.5. *For $f \in \mathcal{F}_{\mathcal{P}} L$, the following statements are equivalent.*

(1) $f \in \mathcal{F}_{\mathcal{P}}^{*} L$.

(2) $f(]\!]p, q[\![) = \top$ *for some* $p, q \in \mathbb{Q}$ *with* $p < q$.

(3) $f((-\infty, p[\![\cup]\!]q, +\infty)) = \bot$ *for some* $p, q \in \mathbb{Q}$ *with* $p < q$.

Proof. Straightforward. □

Proposition 4.6. *Let L be an $\mathcal{F}_{\mathcal{P}}$-completely regular frame. Then every minimal ideal of $\mathcal{F}_{\mathcal{P}} L$ consists entirely of bounded functions.*

Proof. Let I be a minimal ideal of $\mathcal{F}_{\mathcal{P}} L$ and $f \in I$. Then there exists an atom element a of L such that $I = \mathfrak{R}_{\mathcal{F}_{\mathcal{P}}}(a)$, by Proposition 3.11. Since $f = f f_a$, we conclude from Proposition 3.3 that $f(]\!]-n, n[\![) = a' \vee f(]\!]-n, n[\![)$, for all $n \in \mathbb{N}$. The maximality of a' insures that $a' = f(]\!]-n, n[\![)$ or $f(]\!]-n, n[\![) = \top$. If $a' = f(]\!] - n, n[\![)$ for all $n \in \mathbb{N}$, then $\top = f(\mathbb{R}) = \bigvee f(]\!] - n, n[\![) = a'$, which is a contradiction. Therefore there exists $n \in \mathbb{N}$ such that $f(]\!] - n, n[\![) = \top$. Hence $f \in \mathcal{F}_{\mathcal{P}}^{*} L$, by Proposition 4.5. □

Corollary 4.7. *If L is an $\mathcal{F}_{\mathcal{P}}$-completely regular frame, then* $\mathrm{Soc}(\mathcal{F}_{\mathcal{P}}L) = \mathrm{Soc}(\mathcal{F}^*_{\mathcal{P}}L)$.

Proof. Since $\mathcal{F}_{\mathcal{P}}$ is a reduced f-ring with bounded inversion, then $\mathrm{Soc}(\mathcal{F}_{\mathcal{P}}L) = \mathrm{Soc}(\mathcal{F}^*_{\mathcal{P}}L)$, by [6, Proposition 3.2] and Proposition 4.6. □

Proposition 4.8. *Let L be an $\mathcal{F}_{\mathcal{P}}$-completely regular frame. If L has a finite number of atoms, then the following statements hold.*

(1) *The residue class ring $\mathcal{F}_{\mathcal{P}}L/\mathrm{Soc}(\mathcal{F}_{\mathcal{P}}L)$ has Property (A).*

(2) *The residue class ring $\mathcal{F}^*_{\mathcal{P}}L/\mathrm{Soc}(\mathcal{F}_{\mathcal{P}}L)$ has Property (A).*

Proof. (1) Let $A = \{a_1, \ldots, a_n\}$ be the set of all atoms in L. Then

$$\mathrm{Soc}(\mathcal{F}_{\mathcal{P}}L) = \sum_{i=1}^{n} \mathfrak{R}_{\mathcal{F}_{\mathcal{P}}}(a_i) = \sum_{i=1}^{n} f_{a_i}\mathcal{F}_{\mathcal{P}}L,$$

by Propositions 3.4 and 3.11. For every $f \in \mathcal{F}_{\mathcal{P}}L$, put $\bar{f} = f + \mathrm{Soc}(\mathcal{F}_{\mathcal{P}}L)$. Let $I = \langle \bar{f}_1, \ldots, \bar{f}_n \rangle$ be a finitely generated ideal of $\mathcal{F}_{\mathcal{P}}L/\mathrm{Soc}(\mathcal{F}_{\mathcal{P}}L)$ entirely of zero-divisors. Consider $f = \sum_{i=1}^{n} f_i^2$. Since $\bar{f} = \sum_{i=1}^{n} \bar{f_i}^2 \in I$, we conclude that there exists $0 < g \in \mathcal{F}_{\mathcal{P}}L$ such that $\bar{f}\bar{g} = 0$ and $\bar{g} \neq 0$. Then $coz(fg) = \bigvee_{j=1}^{r} a_{i_j}$, for some $a_{i_1}, \ldots, a_{i_r} \in A$. Consider $b = \bigwedge_{i=1}^{n} a_i'$. We claim that $0 \neq \bar{f_b}\bar{g} \in Ann(I)$. If $coz(f_b g) \leq \bigvee_{i=1}^{n} a_i$, then

$$coz(f_b g) = coz(f_b g) \wedge \bigvee_{i=1}^{n} a_i = b \wedge coz(g) \wedge \bigvee_{i=1}^{n} a_i = \bot,$$

which follows that $coz(g) \leq z(f_b) = \bigvee_{i=1}^{n} a_i$. Hence $g \in Soc(\mathcal{F}_{\mathcal{P}}L)$, which is a contradiction. Thus $0 \neq \bar{f_b}\bar{g}$.

Let $h = \sum_{i=1}^{n} f_i h_i$. Then

$$\begin{aligned} coz(hf_b g) &= coz(\textstyle\sum_{i=1}^{n} f_i h_i f_b g) \\ &\leq \textstyle\bigvee_{i=1}^{n} coz(|f_i||h_i|f_b g) \\ &\leq \textstyle\bigvee_{i=1}^{n} coz(f_i^2 f_b g) \\ &= coz(\textstyle\sum_{i=1}^{n} f_i^2 f_b g) \\ &= coz(fgf_b) \\ &\leq \textstyle\bigvee_{j=1}^{r} a_{i_j} \wedge \bigwedge_{i=1}^{n} a_i' \\ &= \bot. \end{aligned}$$

Hence $\bar{h}\bar{f}_b\bar{g} = 0$, which implies that $\bar{f}_b\bar{g} \in Ann(I)$. Therefore $\mathcal{F}_{\mathcal{P}}L/\mathrm{Soc}(\mathcal{F}_{\mathcal{P}}L)$ has Property (A).

(2) By a similar argument as used in the proof of item (1), one can prove it. □

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