Filters of Coz($X$)

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Dedicated to Bernhard Banaschewski on the occasion of his 90th birthday

Abstract. In this article we investigate filters of cozero sets for real-valued continuous functions, called coz-filters. Much is known for $z$-ultrafilters and their correspondence with maximal ideals of $C(X)$. Similarly, a correspondence will be established between coz-ultrafilters and minimal prime ideals of $C(X)$. We will further notice various properties of coz-ultrafilters in relation to $P$-spaces and $F$-spaces. In the last two sections, the collection of coz-ultrafilters will be topologized, and then compared to the hull-kernel and the inverse topologies placed on the collection of minimal prime ideals of $C(X)$ and general lattice-ordered groups.

1 Introduction and notation

In parallel to the notion of $z$-filters, we first consider the relationship between the coz-filters and the prime ideals of $C(X)$. In particular, we will show that maximal coz-filters (coz-ultrafilters) have a direct correspondence with the minimal prime ideals of $C(X)$. The inspiration for this comes from the

Keywords: Cozero sets, ultrafilters, minimal prime ideals, $P$-space, $F$-space, inverse topology, $\ell$-groups.

Mathematics Subject Classification [2010]: 46E25, 13A15, 06F15, 54A05, 54D80.

Received: 11 December 2016, Accepted: 12 March 2017

ISSN Print: 2345-5853 Online: 2345-5861

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known relationship between maximal ideals of $C(X)$ and $z$-ultrafilters. This is the content of Section 2.

The focus of Section 3 is topologizing the collection of all $coz$-ultrafilters. In particular, we endow topologies on these collections which are similar to the inverse and the hull-kernel topologies. It is shown that these topological spaces are homeomorphic to spaces of minimal prime ideals of $C(X)$.

In the last section we prove some general results for lattice-ordered groups. In particular, we show that the space of ultrafilters of $G^+$, with respect to the Stone topology, is homeomorphic to $\text{Min}(G)^{-1}$, for an $\ell$-group $G$. Moreover, for a $W$-object $G$, there is a one-to-one correspondence between the ultrafilters of $Coz(G)$ and the ultrafilters of $G^+$.

Throughout this paper $X$ is a Tychonoff space, that is, $X$ is a completely regular Hausdorff space. Also, all rings $R$ are commutative with identity. The ring of real-valued continuous functions on $X$ is denoted by $C(X)$, or just simply $C$. We mention a few definitions that are essential in this article. For the rest of the concepts see [1], [11], [13], and [14].

If $f \in C(X)$, then the zero set of $f$ is $z(f) = \{x \in X : f(x) = 0\}$ and the cozero set of $f$ is $coz(f) = \{x \in X : f(x) \neq 0\}$; $z(f)$ and $coz(f)$ are set-theoretic complements of each other in $X$. Denote the collection of all zero sets of $X$ as $Z(X)$ and the collection of all cozero sets of $X$ as $Coz(X)$. A nonempty subfamily $F \subseteq Coz(X)$ is a $coz$-filter on $X$ if the following holds:

1. $\emptyset \notin F$;

2. if $A, B \in F$, than $A \cap B \in F$; and

3. if $A \in F$, $B \in Coz(X)$, and $A \subseteq B$, then $B \in F$.

Observe that the above definition is the counterpart of the concept of $z$-filter from [11]. A prime $coz$-filter is a $coz$-filter $\mathcal{F}$ such that if $coz(f) \cup coz(g) \in \mathcal{F}$, then $coz(f) \in \mathcal{F}$ or $coz(g) \in \mathcal{F}$. A $coz$-filter $\mathcal{U} \subseteq Coz(X)$ is a $coz$-ultrafilter if whenever $\mathcal{U} \subset \mathcal{F}$, where $\mathcal{F}$ is a $coz$-filter, then $\mathcal{F} = Coz(X)$.

A subset $M$ of a ring $R$ is called a multiplicative set if $1 \in M$, and for all $f, g \in M$, the product $fg \in M$. A ring is called reduced if it has no non-zero nilpotent elements; hence, $C(X)$ is a reduced ring. A proper ideal $P$ of $R$ is called a prime ideal if $ab \in P$ implies $a \in P$ or $b \in P$. A minimal prime
ideal of a ring is a prime ideal which contains no other smaller prime ideal. The collection of all minimal prime ideals of $R$ is denoted by $\text{Min}(R)$.

The following results are well-known and will be used often in this article.

Lemma 1.1. If $f \in C(X)$, then the following are equivalent:

1. $f$ is a non-zero-divisor of $C(X)$, that is, $\text{Ann}(f) = 0$.
2. $\text{coz}(f)$ is dense in $X$, that is, $\text{cl}(\text{coz}(f)) = X$.
3. $\text{z}(f)$ has empty interior, that is, $\text{int}(\text{z}(f)) = \emptyset$.

Similarly, notice that given $f \in C(X)$, $f = 0$ precisely when $\text{coz}(f)$ is an empty set; this is equivalent to $\text{z}(f) = X$.

For more on Lemma 1.1 refer to Section 5 in [7] and Theorem 2.2 in [9].

Proposition 1.2. Let $U$ be a $\text{coz}$-filter. Then $U$ is a $\text{coz}$-ultrafilter if and only if given any $\text{coz}(f) \notin U$, there exists some $\text{coz}(g) \in U$ such that $fg = 0$ (equivalently, $\text{coz}(f) \cap \text{coz}(g) = \emptyset$). As a consequence, $\text{coz}(f)$ is dense in $X$ if and only if $\text{coz}(f) \in U$ for every $\text{coz}$-ultrafilter $U$.

It is known that an ultrafilter is a prime filter. This is true for $\text{coz}$-ultrafilters as well. The next result follows from the preceding proposition.

Corollary 1.3. If $U$ is a $\text{coz}$-ultrafilter, then it is a prime $\text{coz}$-filter.

2 Prime ideals and $\text{coz}$-filters

Gillman and Jerison devoted the second chapter of their text *Rings of Continuous Functions* to the relationship between the ideals of $C$ and the $\text{z}$-filters. Here we provide a few of their results as a reference for the inspiration of the results which follow:

Theorem 2.1 (Theorem 2.3, [11]). (1) If $I$ is an ideal in $C$, then $Z(I) = \{z(f) : f \in I\}$ is a $\text{z}$-filter on $X$.

(2) If $\mathcal{F}$ is a $\text{z}$-filter on $X$, then $Z^{-}(\mathcal{F}) = \{f : z(f) \in \mathcal{F}\}$ is an ideal in $C$. 

If \( I \) is an ideal of \( C \), then \( I \) is a \( z \)-ideal if \( z(f) \in Z(I) \) implies \( f \in I \), that is to say if \( I = Z^{\leftarrow}(Z(I)) \). [Note, \( F = Z(Z^{\leftarrow}(F)) \) is always true for a \( z \)-filter \( F \).]

With \( z \)-ideals more can be said about the relationship between the prime ideals of \( C \) and the prime \( z \)-filters.

**Theorem 2.2** (Theorem 2.12, [11]).

1. If \( P \) is a prime ideal of \( C \), then \( Z(P) \) is a prime \( z \)-filter.
2. If \( F \) is a prime \( z \)-filter, then \( Z^{\leftarrow}(F) \) is a prime \( z \)-ideal.

Furthermore, it has been shown that the maximal ideals of \( C \) are in one-to-one correspondence to the \( z \)-ultrafilters on \( X \).

The results from Gillman and Jerison are well-known, but the relationship between the ideals of \( C \) and the \( coz \)-filters on \( X \) are yet to be considered; this section is devoted to such an investigation. Through the results in this section we notice the difference between the \( z \)-filters and the \( coz \)-filters; to start, observe that we require the ideal of \( C \) to be prime to obtain a \( coz \)-filter of \( Coz(X) \).

If \( P \) is an ideal of \( C \), then \( P^c \) is the set-theoretic complement of \( P \) in \( C \).

**Theorem 2.3.** If \( P \) is a prime ideal of \( C(X) \), then \( Coz(P^c) = \{ coz(f) : f \in P^c \} \) is a \( coz \)-filter.

**Proof.** The first two axioms of being a \( coz \)-filter can be easily verified. To show the third axiom we will require the primeness of \( P \). Suppose \( coz(f) \subseteq coz(g) \) for some \( f \notin P \) and \( g \in C(X) \). Since \( P \) is prime, using 5D in [11], it follows that \( f^2 + g^2 \in P^c \). Therefore \( coz(g) \in Coz(P^c) \), since \( coz(g) = coz(f^2 + g^2) \).

This result is different than the \( Z(I) \) case, since \( I \) need only be an ideal for \( Z(I) \) to be a \( z \)-filter. Primeness is needed since this is related to a multiplicative set which will become evident. The following example will demonstrate this fact.

**Example 2.4.** Let \( C = C(\mathbb{N}) \), where \( \mathbb{N} \) is the natural numbers with the discrete topology. Recall that the characteristic function on \( S \subseteq \mathbb{N} \) is \( \chi_S \).

In this case, every characteristic function is continuous, since \( \mathbb{N} \) is discrete. Now, consider \( I = \langle \chi_{\{1\}} \rangle \), the ideal generated by \( \chi_{\{1\}} \). Then \( z(\chi_{\{1\}}) = \mathbb{N}\setminus\{1\} \) and for any \( f \in I \), \( z(\chi_{\{1\}}) \subseteq z(f) \). It follows that \( \chi_{\{2\}} \) and \( \chi_{\{3\}} \)
are not in \( I \). So, \( \text{coz}(\chi_{\{2\}}), \text{coz}(\chi_{\{3\}}) \in \text{Coz}(I^c) \). However, \( \text{coz}(\chi_{\{2\}}) \cap \text{coz}(\chi_{\{3\}}) = \{2\} \cap \{3\} = \emptyset \). Therefore, \( \text{Coz}(I^c) \) is not a \( \text{coz} \)-filter. Do note that \( \chi_{\{1,2\}} \chi_{\{1,3\}} = \chi_{\{1\}} \in I \), but \( \chi_{\{1,2\}}, \chi_{\{1,3\}} \notin I \), so \( I \) is not prime.

We know that there is a one-to-one correspondence between the \( z \)-ideals of \( C \) and the \( z \)-filters of \( X \). Such a correspondence does not exist for \( \text{coz} \)-filters. In order to establish a relation between \( z \)-ideals and \( \text{coz} \)-filters, we need the prime condition on both. The next result is a counterpart of Theorem 2.12 in [11]; there is a one-to-one correspondence between the prime \( z \)-ideals of \( C \) and the prime \( \text{coz} \)-filters of \( X \).

**Theorem 2.5.** If \( P \) is a prime \( z \)-ideal of \( C \), then \( \text{Coz}(P^c) \) is a prime \( \text{coz} \)-filter. If \( \mathcal{F} \) is a prime \( \text{coz} \)-filter, and \( M = \text{Coz}^-(\mathcal{F}) = \{ f : \text{coz}(f) \in \mathcal{F} \} \), then \( P = C \setminus M \) is a prime \( z \)-ideal.

Moreover, \( \text{Coz}^-(\text{Coz}(P^c)) = P^c \) and \( \text{Coz}(\text{Coz}^-(\mathcal{F})) = \mathcal{F} \), for all prime \( z \)-ideals \( P \) and all prime \( \text{coz} \)-filters \( \mathcal{F} \).

**Proof.** The proof of the first part is left to the interested reader.

We show that \( P \) is a prime \( z \)-ideal. To show that \( P \) is an ideal, let \( f, g \in P \). Then \( \text{coz}(f), \text{coz}(g) \notin \mathcal{F} \). This implies that \( \text{coz}(f) \cup \text{coz}(g) \notin \mathcal{F} \), since \( \mathcal{F} \) is prime. Note that \( \text{coz}(f - g) \subseteq \text{coz}(f) \cup \text{coz}(g) \). Since \( \mathcal{F} \) is \( \text{coz} \)-filter, \( \text{coz}(f - g) \notin \mathcal{F} \). Hence \( f - g \notin M \); in other words, \( f - g \notin P \). Now let \( f \in P \) and \( h \in C \). Since \( \text{coz}(f) \notin \mathcal{F} \) and \( \text{coz}(fh) \subseteq \text{coz}(f) \), therefore \( \text{coz}(fh) \notin \mathcal{F} \). Consequently, \( fh \notin P \). Furthermore, \( P \) is a proper ideal since given any unit \( u \in C \), \( \text{coz}(u) = X \in \mathcal{F} \), which means \( P \) does not contain any unit. Observe that for all \( f, g \in C \), \( fg \notin P \) whenever \( f, g \notin P \), proving that \( P \) is prime. Thus \( P \) is a \( z \)-ideal by the definition of \( M \).

Finally, it is easy to check that \( \text{Coz}(\text{Coz}^-(\mathcal{F})) = \mathcal{F} \), for all prime \( \text{coz} \)-filters \( \mathcal{F} \). On the other hand, if \( f \in \text{Coz}^-(\text{Coz}(P^c)) \), then \( \text{coz}(f) = \text{coz}(g) \) for some \( g \notin P \). Since \( P \) is a \( z \)-ideal, \( f \in P^c \). Thus, \( \text{Coz}^-(\text{Coz}(P^c)) \subseteq P^c \). The other inclusion can be verified easily. Hence the result follows.

The following example shows that the prime condition of a \( \text{coz} \)-filter is required in the preceding theorem.

**Example 2.6.** Consider \( C(\mathbb{N}) \) and let \( \mathcal{F} = \{ A \subseteq \mathbb{N} : \{2n : n \in \mathbb{N}\} \subseteq A \} \), this is the collection of all the subsets of \( \mathbb{N} \) which contain all of the even natural numbers. It is clear that \( \mathcal{F} \) is a \( \text{coz} \)-filter since all subsets of \( \mathbb{N} \) are
clopen. Next, take \( \{2, 4, 6, 8, \ldots \} \subseteq \mathcal{F} \), this is equal to \( \{2\} \cup \{4, 6, 8, \ldots \} \).
However, both \( \{2\} \) and \( \{4, 6, 8, \ldots \} \) are not in \( \mathcal{F} \), so \( \mathcal{F} \) is not a prime \( \text{coz} \)-filter.

Next, take \( M = \{ f : \text{coz}(f) \in \mathcal{F} \} \). Since \( \{2\} \) and \( \{4, 6, 8, \ldots \} \) are not in \( \mathcal{F} \), \( \chi_{\{2\}} \) and \( \chi_{\{4, 6, 8, \ldots \}} \) are not in \( M \), hence they are in \( C \setminus M \). Further, we see that \( \text{coz} (\chi_{\{2\}} - \chi_{\{4, 6, 8, \ldots \}}) = \{2n : n \in \mathbb{N}\} \), this implies that \( \chi_{\{2\}} - \chi_{\{4, 6, 8, \ldots \}} \in M \) and not in \( C \setminus M \). So, \( C \setminus M \) is not an ideal.

Note that the set \( M \), in Theorem 2.5, is a multiplicative set. Recall some general results on multiplicative sets which will be used later, but will not be referenced to directly.

**Theorem 2.7** (Chapter 3, Example 1, [1]; Page 2, Theorem 2, [14]).

1. If \( P \) is a prime ideal of \( C \), then \( C \setminus P \) is a multiplicative set.
2. \( M \subseteq C \) is a maximal multiplicative set if and only if \( C \setminus M \) is a minimal prime ideal of \( C \).

For more on general multiplicative sets refer to [1] or [16].

Next, we consider \( \text{coz} \)-ultrafilters and how they relate to prime ideals. Gillman and Jerison provide us with the following:

**Theorem 2.8** (Theorem 2.5, [11]). If \( M \) is a maximal ideal in \( C \), then \( Z(M) \) is a \( z \)-ultrafilter on \( X \).

We will establish a correspondence between the minimal prime ideals of \( C \) and the \( \text{coz} \)-ultrafilters; or, another way to view the situation is that there is a relationship between the maximal multiplicative sets and the \( \text{coz} \)-ultrafilters. Recall the following known results:

**Theorem 2.9** (Theorem 14.7, [11]). Every minimal prime ideal of \( C \) is a \( z \)-ideal.

**Lemma 2.10** (Corollary 2.2, [13]). Let \( R \) be a reduced ring and let \( P \) be a prime ideal of \( R \). Then \( P \) is minimal if and only if for each \( x \in P \) there exists an \( r \in R \setminus P \) such that \( xr = 0 \).

Now we state the main theorem for this section.

**Theorem 2.11.** (1) If \( U \) is a \( \text{coz} \)-ultrafilter of \( X \) and \( M = \{ f : \text{coz}(f) \in U \} \) with \( P = C \setminus M \), then \( P \) is a minimal prime ideal.
(2) If \( P \) is a minimal prime ideal of \( C \), then \( \text{Coz}(P^c) \) is a \( \text{coz} \)-ultrafilter.
Proof. (1) Suppose $U$ is a coz-ultrafilter on $X$. By Theorems 1.3 and 2.5, it follows that $P$ is a prime $z$-ideal. To show that $P$ is a minimal prime ideal, let $f \in P$. Since $f \notin M$, it follows that $\text{coz}(f) \notin U$. Using Proposition 1.2, there exists some $g \in C(X)$ such that $\text{coz}(g) \in U$ and $fg = 0$. Consequently, by Lemma 2.10, $P$ is minimal.

(2) Suppose $P$ is a minimal prime ideal of $C$ and let $\text{coz}(g) \notin \text{Coz}(P^c)$. It follows that $g \in P$. So, by Lemma 2.10, there exists $t \in \text{Ann}(g)$ with $t \notin P$. Consequently, $\text{coz}(t) \cap \text{coz}(g) = \text{coz}(tg) = \text{coz}(0) = \emptyset$. Since $t \notin P$, we have $\text{coz}(t) \in \text{Coz}(P^c)$. By Proposition 1.2, $\text{Coz}(P^c)$ is a coz-ultrafilter. □

The remainder of the section will focus on coz-filters related to $P$-spaces and $F$-spaces. The inspiration for the following investigation is given by the sets $A_p = \{ z(f) : p \in z(f) \}$, defined in [11]. Note that $A_p$ is a $z$-filter for every $p \in X$.

In a similar flavor we define $B_p$ to be the collection of all cozero neighborhoods of $p$, that is, $B_p = \{ \text{coz}(f) : p \in \text{coz}(f) \}$. Observe that $B_p$ is a prime coz-filter with $\bigcap B_p = \{ p \}$. On the other hand, $B_p$ is not necessarily a coz-ultrafilter. For an example, consider $\mathbb{R}$ with the usual topology and $1 \in \mathbb{R}$. Then $B_1$ is not a coz-ultrafilter, since the cozero set $(-\infty, 1)$ intersects every member of $B_1$.

Recall that for $p \in X$, $M_p = \{ f \in C : f(p) = 0 \}$ is a maximal ideal and $O_p = \{ f \in C : p \in \text{int}(z(f)) \}$ is a $z$-ideal, with $O_p \subseteq M_p$. Refer to 4.6-4.7 and 4I in [11] for more on $M_p$ and $O_p$. Further, if $M_p = O_p$, then $p$ is called a $P$-point. We say that $X$ is a $P$-space if every element of $X$ is a $P$-point (refer to 4L.2 in [11]).

Observe that for $p \in X$,

$$\text{Coz}^c(B_p) = C \setminus M_p \text{ and } \text{Coz}(C \setminus M_p) = B_p.$$ 

Hence, using Theorem 2.11 and 4I in [11], we have the following characterization of when $B_p$ is a coz-ultrafilter.

**Theorem 2.12.** Let $p \in X$. Then $B_p$ is a coz-ultrafilter if and only if $p$ is a $P$-point. Further, $B_p$ is a coz-ultrafilter for all $p \in X$ if and only if $X$ is a $P$-space.

Following the pattern of the nomenclature, we define $C_p = \{ \text{coz}(f) : p \in \text{cl}(\text{coz}(f)) \}$. Note that $B_p \subseteq C_p$ for all $p \in X$. In general, $C_p$ is not a...
coz-filter. For example, consider $C(R)$. We have $0 \in [-1, 0] = \text{cl}((-1, 0))$ and $0 \in [0, 1] = \text{cl}((0, 1))$, and so $(-1, 0), (0, 1) \in C_0$. However, $\text{cl}((-1, 0) \cap (0, 1)) = \text{cl}(\emptyset) = \emptyset$, which is not in $C_0$.

Observe that $C_p$ satisfies the prime condition: $p \in \text{cl}(\text{coz}(f) \cup \text{coz}(g)) = \text{cl}(\text{coz}(f)) \cup \text{cl}(\text{coz}(g))$. Consequently, $p \in \text{cl}(\text{coz}(f))$ or $p \in \text{cl}(\text{coz}(g))$.

Our next result gives a characterization of when $C_p$ is a coz-filter.

**Proposition 2.13.** Let $p \in X$. Then $C_p$ is a prime coz-filter if and only if $O_p$ is a prime ideal.

**Proof.** ($\Rightarrow$) Suppose $fg \in O_p$. Then $p \in \text{int}(z(fg))$, and so $p \notin \text{cl}(\text{coz}(fg))$. Consequently $\text{coz}(fg) \notin C_p$, which gives $\text{coz}(f) \cap \text{coz}(g) \notin C_p$. So, $\text{coz}(f) \notin C_p$ or $\text{coz}(g) \notin C_p$, since $C_p$ is a coz-filter. Hence $p \notin \text{cl}(\text{coz}(f))$ or $p \notin \text{cl}(\text{coz}(g))$. So, $p \in \text{int}(z(f))$ or $p \in \text{int}(z(g))$, and therefore $f \in O_p$ or $g \in O_p$, that is, $O_p$ is prime.

($\Leftarrow$) It is evident that $\emptyset \notin C_p$. If $\text{coz}(f), \text{coz}(g) \in C_p$, then $p \notin \text{int}(z(f))$ and $p \notin \text{int}(z(g))$. Consequently, $f, g \notin O_p$. Since $O_p$ is prime, $fg \notin O_p$. Thus, $p \notin \text{int}(z(fg))$. It follows that $p \in \text{cl}(\text{coz}(fg)) = \text{cl}(\text{coz}(f) \cap \text{coz}(g))$, and so $C_p$ is closed under intersection. Lastly, let $\text{coz}(f) \in C_p$ and $\text{coz}(g) \in \text{Coz}(X)$, with $\text{coz}(f) \subseteq \text{coz}(g)$. Since $\text{cl}(\text{coz}(f)) \subseteq \text{cl}(\text{coz}(g))$, we have that $\text{coz}(g) \in C_p$. $\square$

Recall that a space $X$ is called an $F'$-space if $O_p$ is a prime ideal, for all $p \in X$. A space $X$ is called an $F$-space if every finitely generated ideal of $C$ is a principal ideal. An $F$-space is also an $F'$-space, and the two notions are equivalent if $X$ is compact. Equivalent conditions for $X$ to be an $F$-space or an $F'$-space are well-known (see [10] and [11]). In the following theorem we provide some more equivalent conditions for $F'$-spaces in terms of $C_p$.

Recall that when $O_p$ is a prime ideal, it is a minimal prime ideal of $C$. Moreover, $\text{Coz}(C \setminus O_p) = C_p$ and $\text{Coz}^c(C_p) = C \setminus O_p$. Using these facts and Theorem 2.11, the following theorem holds.

**Theorem 2.14.** For a (compact) space $X$, the following are equivalent:

1. $X$ is an $F'$-space ($F$-space).
2. $C_p$ is a prime coz-filter, for all $p \in X$.
3. $C_p$ is a coz-ultrafilter, for all $p \in X$. 

3 The spaces of coz-ultrafilters

In this section, we investigate the relationship between the spaces of \( \text{Min}(C(X)) \), with respect to the hull-kernel and the inverse topologies, and the spaces of coz-ultrafilters.

Let \( \mathcal{U}(X) \) denote the collection of all coz-ultrafilters of \( X \). Let \( \Phi : \text{Min}(C(X)) \to \mathcal{U}(X) \) be defined by \( \Phi(P) = \text{Coz}(P^c) = \{ \text{coz}(f) : f \in P^c \} \). Clearly \( \Phi \) is a bijection, implied by Theorem 2.11.

Given any \( f \in C(X) \), let us use the conventional notation \( V_\mathcal{U}(f) = \{ F \in \mathcal{U}(X) : \text{coz}(f) \in F \} \). The set-theoretic complement of \( V_\mathcal{U}(f) \) is denoted by \( U_\mathcal{U}(f) \).

**Proposition 3.1.** For \( f,g \in C \) we have:

1. \( V_\mathcal{U}(f) \cap V_\mathcal{U}(g) = V_\mathcal{U}(fg) \) and \( U_\mathcal{U}(f) \cup U_\mathcal{U}(g) = U_\mathcal{U}(f^2 + g^2) \).
2. \( V_\mathcal{U}(f) \cup V_\mathcal{U}(g) = V_\mathcal{U}(f^2 + g^2) \) and \( U_\mathcal{U}(f) \cap U_\mathcal{U}(g) = U_\mathcal{U}(f^2 + g^2) \).
3. \( f \in P \in \text{Min}(C(X)) \) if and only if \( f \in \bigcap \text{Min}(C(X)) = 0 \).
4. \( V_\mathcal{U}(f) = \mathcal{U}(X) \) if and only if \( f \) is a non-zero-divisor if and only if \( U_\mathcal{U}(f) = \emptyset \).

**Proof.** (1) This is due to the fact that given any coz-filter \( F, \text{coz}(f), \text{coz}(g) \in F \) if and only if \( \text{coz}(fg) \in F \).

(2) This is due to the fact that given any prime coz-filter \( F, \text{coz}(f) \cup \text{coz}(g) \in F \) if and only if \( \text{coz}(f) \in F \) or \( \text{coz}(g) \in F \).

If \( V_\mathcal{U}(f) = \emptyset \), then \( \text{coz}(f) \notin F \) for every \( F \in \mathcal{U}(X) \). Consequently, \( f \in P \) for every \( P \in \text{Min}(C(X)) \) and hence \( f \in \bigcap \text{Min}(C(X)) = 0 \). The other direction is clear.

(3) This follows directly from Lemma 1.1 and Proposition 1.2. \( \square \)

Based on the properties described above, we can talk about two different topologies on \( \mathcal{U}(X) \):

(i) the collection \( \{ V_\mathcal{U}(f) : f \in C \} \) forms a basis for open sets on \( \mathcal{U}(X) \), this topology is denoted by \( \mathcal{U}(X)^{-1} \).

(ii) the collection \( \{ U_\mathcal{U}(f) : f \in C \} \) also forms a basis for open sets on \( \mathcal{U}(X) \). We use \( \mathcal{U}(X) \) to denote this topology.

\( \mathcal{U}(X) \) is the same topology considered by Wallman [19], Samuel [18], and Brooks [4], generated by the collection \( \{ V_\mathcal{U}(f) : f \in C \} \) as a basis for...
closed sets. It is known that \( \mathcal{U}(X) \) is a compact \( T_1 \) space. We show a more interesting fact about \( \mathcal{U}(X) \) with a different viewpoint of this topology.

These two topological spaces on \( \mathcal{U}(X) \) have similar flavors as the spaces of \( \text{Min}(C(X)) \). In [12] the concept of the hull-kernel topology, also referred to as the Stone topology, for \( \text{Min}(R) \) is studied for commutative rings. Recall that the hull-kernel topology on \( \text{Min}(C(X)) \), denoted by \( \text{Min}(C(X)) \), has subsets of the form \( U(f) = \{ P \in \text{Min}(C(X)) : f \notin P \} \) as basic open sets, for \( f \in C \). It is well-known that \( \text{Min}(C(X)) \) is a zero dimensional, Hausdorff topological space (see Corollary 2.4 in [12]).

On the other hand, the inverse topology on \( \text{Min}(C(X)) \), denoted by \( \text{Min}(C(X))^{-1} \), is generated by the subsets of the form

\[
V(f) = \{ P \in \text{Min}(C(X)) : f \in P \}
\]

as basic open sets, for \( f \in C \). The inverse topology on \( \text{Min}(G) \) is considered in [17] for a lattice-ordered group \( G \). Also, \( \text{Min}(C(X))^{-1} \) is studied in [3]; \( \text{Min}(C(X))^{-1} \) is a compact, \( T_1 \) space.

**Theorem 3.2.** The following hold for any Tychonoff space \( X \):

1. \( \mathcal{U}(X)^{-1} \) is homeomorphic to \( \text{Min}(C(X)) \).
2. \( \mathcal{U}(X) \) is homeomorphic to \( \text{Min}(C(X))^{-1} \).

**Proof.** Consider \( \Phi : \text{Min}(C(X)) \rightarrow \mathcal{U}(X) \) as defined at the beginning of Section 4. We have noticed earlier that \( \Phi \) is a well-defined bijection.

(1) For \( f \in C \), we claim that \( \Phi^{-1}(V_{\mathcal{U}}(f)) = U(f) \). Consider \( P \in \Phi^{-1}(V_{\mathcal{U}}(f)) \), then \( \text{Coz}(P^c) \in V_{\mathcal{U}}(f) \). It follows that \( \text{coz}(f) \in \text{Coz}(P^c) \), and so \( \text{coz}(f) = \text{coz}(g) \) for some \( g \in P^c \). If \( f \in P \), then \( g \in P \) since \( P \) is a \( z \)-ideal. Therefore, \( f \notin P \) which means \( P \in U(f) \). Conversely suppose \( P \in U(f) \), that is \( f \notin P \). Since \( \text{coz}(f) \in \text{Coz}(P^c) \), we have \( \Phi(P) = \text{Coz}(P^c) \in V_{\mathcal{U}}(f) \). Accordingly, \( P \in \Phi^{-1}(V_{\mathcal{U}}(f)) \). Hence, \( \Phi : \text{Min}(C(X)) \rightarrow \mathcal{U}(X)^{-1} \) is a continuous function.

Finally we show that \( \Phi \) is an open map. Let \( \Phi(Q) \in \Phi(U(f)) \subseteq \mathcal{U}(X)^{-1} \) for some \( Q \in U(f) \). Since \( f \notin Q \), we have \( \text{coz}(f) \in \text{Coz}(Q^c) \), which means that \( \Phi(Q) \in V_{\mathcal{U}}(f) \). Now, let \( F \in V_{\mathcal{U}}(f) \) be arbitrary. Then there exists some \( P \in \text{Min}(C(X)) \) such that \( F = \Phi(P) = \text{Coz}(P^c) \). Note that \( \text{coz}(f) \in F \) implies that \( f \notin P \), since \( P \) is a \( z \)-ideal. Consequently \( P \in U(f) \), and so \( F = \Phi(P) \in \Phi(U(f)) \).

(2) The homeomorphic aspect of \( \Phi : \text{Min}(C(X))^{-1} \rightarrow \mathcal{U}(X) \) can be shown similarly and is left to the interested reader.
Corollary 3.3. $\mathcal{U}(X)^{-1}$ is a zero-dimensional, Hausdorff topological space.

We direct our attention to [17] where it is shown that if $X$ is an $F$-space, then $\text{Min}(C(X))^{-1}$ is homeomorphic to $\beta(X)$, the Stone-Čech compactification of $X$. Hence, we have the following result:

Corollary 3.4. If $X$ is a compact $F$-space, then $\mathcal{U}(X)$ is homeomorphic to $X$.

Definition 3.5 (Definition 2.3, [4]). Let $L$ be a lattice on $X$. $L$ is a normal lattice if for each pair of disjoint elements $A$ and $B$ of $L$, there exists $C, D \in L$ such that $A \subseteq C$, $B \subseteq D$, $A \cap D = \emptyset = B \cap C$, and $C \cup D \in U$ for every ultrafilter $U$ of $L$.

In [4], it is shown that the collection of ultrafilters, with the Samuel topology, on a lattice $L$ is Hausdorff precisely when $L$ is a normal lattice. Here we are considering $\text{Coz}(X)$ as a lattice on the set $\mathcal{P}(X)$. In [3] the authors have shown various conditions for the space $\text{Min}(C(X))^{-1}$ to be Hausdorff. A further study on the inverse topology on the minimal prime subgroups of lattice-ordered groups is being conducted currently in [2]. Since we have established a homeomorphism between $\mathcal{U}(X)$ and $\text{Min}(C(X))^{-1}$, summing up, we can say the following:

Proposition 3.6. The following are equivalent:

1. $\mathcal{U}(X)$ is a Hausdorff space.
2. $\text{Min}(C(X))^{-1}$ is a Hausdorff space.
3. For each pair of disjoint cozero sets $\text{coz}(f)$ and $\text{coz}(g)$, there exist $f_1, g_1 \in C$ such that $\text{coz}(f) \subseteq \text{int}(z(f_1))$, $\text{coz}(g) \subseteq \text{int}(z(g_1))$, and $\text{int}(z(f_1)) \cap \text{int}(z(g_1)) = \emptyset$.
4. For each pair of disjoint cozero sets $C, D \subseteq X$, there exist cozero sets $C_1, D_1 \subseteq X$ such that $C \subseteq C_1$, $D \subseteq D_1$, $C \cap D_1 = \emptyset$, $D \cap C_1 = \emptyset$, and $C_1 \cup D_1$ is a dense subset of $X$.
5. $\text{Coz}(X)$ is a normal lattice.

Proof. The equivalence of (1) and (2) follows from Theorem 3.2. That (2), (3), and (4) are equivalent is shown in [3]. In [4] it is shown that (1) and (5) are equivalent. We want to point out further that a direct equivalence of (4) and (5) can be shown using Proposition 1.2.
4 Lattice-ordered groups

We finish our article with some general results on ultrafilters associated with lattice-ordered groups. These are generalizations of results from Sections 2 and 3. As a reference for the theory of lattice-ordered groups, see [6].

A lattice-ordered group or \( \ell \)-group is a group \((G, +, 0)\) which is also a lattice \((G, \leq)\) such that for every \(a, b, x, y \in G\), \(a \leq b\) implies that \(x + a + y \leq x + b + y\). In general, an \(\ell\)-group is not necessarily abelian, but do note that \(C\) is an abelian \(\ell\)-group. The positive cone of \(G\) is denoted by \(G^+ = \{g \in G : g \geq 0\}\). An \(\ell\)-subgroup of \(G\) is a subgroup which is also a sublattice. An \(\ell\)-subgroup \(H\) of \(G\) is convex if whenever \(a \in G\) and \(0 \leq a \leq h\) for some \(h \in H^+\), then \(a \in H\). Given \(g \in G\), the set \(g^+ = \{h \in G : |g| \wedge |h| = 0\}\) is called the polar of \(g\). It turns out that polars of \(G\) are convex \(\ell\)-subgroups.

We say an element \(g \in G\) is a weak order unit if \(g^+ = 0\).

Recall the well-known Lemma on Ultrafilters where a one-to-one correspondence has been shown between the minimal prime subgroups of \(G\), \(\text{Min}(G)\), and the ultrafilters of \(G^+\), denoted as \(\text{ult}(G^+)\).

**Lemma 4.1** (Lemma on Ultrafilters, [5]). Let \(G\) be an \(\ell\)-group. For each minimal prime subgroup \(P\), the set \(u(P) = \{q > 0 : q \notin P\}\) is an ultrafilter. Conversely, if \(U\) is an ultrafilter, then \(Q(U) = \bigcup\{x^+ : x \in U\}\) is a minimal prime subgroup. The correspondences \(P \to u(P)\) and \(U \to Q(U)\) are mutually inverse bijections.

We use \(\Psi : \text{Min}(G) \to \text{ult}(G^+)\) to be the bijective map described above, that is, \(\Psi(P) = u(P) = G^+ \setminus P\). Given \(g \in G^+\), let \(N(g) = \{U \in \text{ult}(G^+) : g \in U\}\) be in the power set of \(\text{ult}(G^+)\). Note the following properties of \(N(g)\):

1. If \(g\) is a weak order unit, then \(g \notin P\) for all \(P \in \text{Min}(G)\). Consequently, \(N(g) = \text{ult}(G^+)\).
2. \(N(0) = \emptyset\), since \(0 \notin P\) for all \(P \in \text{Min}(G)\).
3. \(N(g) \cap N(h) = N(g \wedge h)\), since for any \(P \in \text{Min}(G)\), \(g \wedge h \in P\) if and only if \(g \in P\) or \(h \in P\).
4. \(N(g) \cup N(h) = N(g \vee h)\): If \(g \notin U\) and \(h \notin U\), then there exists \(a, b \in U\) such that \(a \wedge g = 0\) and \(b \wedge h = 0\). So, \((g \vee h) \wedge (a \wedge b) = (g \wedge a \wedge b) \vee (h \wedge a \wedge b) = 0\). Since \(a \wedge b \in U\), it follows that \(g \vee h \notin U\).
The collection \( \{N(g) : g \in G^+\} \) forms a closed base for a topology on \( \ult(G^+) \). Further, the inverse topology on \( \Min(G) \), denoted by \( \Min(G)^{-1} \), is defined in a similar manner as to that of \( \Min(C(X))^{-1} \): the collection \( \{V(g) : g \in G^+\} \) forms an open base for the topology, where \( V(g) = \{P \in \Min(G) : g \in P\} \). The topological space \( \Min(G)^{-1} \) is compact \( T_1 \).

**Theorem 4.2.** The map \( \Psi : \Min(G)^{-1} \to \ult(G^+) \) is a homeomorphism. In particular, \( \Psi(V(g)) = \ult(G^+) \setminus N(g) \) for all \( g \in G^+ \).

**Proof.** We only need to show that \( \Psi(V(g)) = \ult(G^+) \setminus N(g) \) for all \( g \in G^+ \). Then using the fact that \( \Psi \) is a bijective map, it follows immediately that \( \Psi \) is continuous and hence a homeomorphism.

Let \( V(g) \) be a basic open set of \( \Min(G)^{-1} \) and \( G^+ \setminus Q = \Psi(Q) \in \Psi(V(g)) \) for some \( Q \in V(g) \). Since \( g \in Q \), \( g \notin G^+ \setminus Q \in \ult(G^+) \). Therefore \( \Psi(Q) = G^+ \setminus Q \in \ult(G^+) \setminus N(g) \). On the other hand, if \( U \in \ult(G^+) \setminus N(g) \), then \( g \notin U = G^+ \setminus P \) for some \( P \in \Min(G)^{-1} \), and so \( g \in P \). Hence \( P \in V(g) \) and \( U = \Psi(P) \in \Psi(V(g)) \).

**Corollary 4.3.** The space \( \ult(C(X)^+) \) is homeomorphic to \( \Min(C(X))^{-1} \).

We next consider the category \( \mathbf{W} \) whose objects are archimedean \( \ell \)-groups with a designated weak order unit and whose morphisms are the \( \ell \)-homomorphisms which preserves the designated unit. An \( \ell \)-group \( G \) is **archimedean** if whenever \( g, h \in G^+ \) and \( ng \leq h \) for all \( n \in \mathbb{N} \), then \( g = 0 \). An archimedean \( \ell \)-group is abelian. Let \( (G, u) \in \mathbf{W} \). Using Zorn’s Lemma, there exist convex \( \ell \)-subgroups which are maximal with respect to not containing \( u \). Such a convex \( \ell \)-subgroup is called a **value** of \( u \), and the collection of all such values is denoted by \( YG \). We endow \( YG \) with the hull-kernel topology, which is a compact Hausdorff space. Let \( \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm \infty\} \) be the two-point compactification of \( \mathbb{R} \). For a topological space \( X \), we define

\[
D(X) = \{f : X \to \overline{\mathbb{R}} : f \text{ is continuous and } f^{-1}(\mathbb{R}) \text{ is dense}\}.
\]

Then \( D(X) \) is a lattice under pointwise operation, but not necessarily a group. We say \( H \subseteq D(X) \) is an \( \ell \)-subgroup of \( D(X) \) if \( H \) is an \( \ell \)-group under pointwise operation. As an example, \( C(X) \subseteq D(X) \) is an \( \ell \)-subgroup. Given an \( f \in D(X) \), we write \( \coz(f) = \{x \in X : f(x) \neq 0\} \), and is called the **cozero-set of** \( f \). The set \( z(f) = X \setminus \coz(f) \) is called the **zero-set of** \( f \).
**Theorem 4.4** (The Yosida Embedding Theorem). Let \((G,u) \in W\). Then there is an \(\ell\)-isomorphism \((g \mapsto \hat{g})\) of \(G\) onto an \(\ell\)-subgroup \(\hat{G}\) of \(D(YG)\) such that \(\hat{u} = 1\) and \(\hat{G}\) has the following separation property: for each \(p \in YG\) and closed set \(V \subseteq YG\) not containing \(p\), there is some \(g \in G\) for which \(\hat{g}(p) = 1\) and \(\hat{g}(V) = 0\). Further, \(YG\) is the unique compact space, up to homeomorphism, that satisfies these two properties.

We identify \(g \in G\) with \(\hat{g} \in D(YG)\). Note that \(\{\text{czo}(g) : g \in G\}\) forms an open basis for the topology of \(YG\); any subset of \(YG\) of this form is called a \(G\)-cozero-set, and the collection of all such is denoted by \(\text{Coz}(G)\). We denote the collection of all cozero sets of \(YG\) by \(\text{Coz}(YG)\). In general, \(\text{Coz}(G) \subseteq \text{Coz}(YG)\). We use \(\mathcal{U}(YG)\) and \(\mathcal{U}(G)\) to denote the ultrafilters on \(\text{Coz}(YG)\) and \(\text{Coz}(G)\), respectively. Using Theorem 3.2, we can immediately conclude that \(\mathcal{U}(YG)\) is homeomorphic to \(\text{Min}(C(YG))^{-1}\), since \(YG\) is a Tychonoff space.

Next, we focus on \(\text{Coz}(G)\)-ultrafilters. Let \(\Omega : \text{ult}(G^+) \to \mathcal{U}(G)\) be defined by \(\Omega(F) = \text{Coz}(F) = \{\text{czo}(f) : f \in F\}\).

**Lemma 4.5.** The map \(\Omega\) defined above is well-defined.

**Proof.** We first show that \(\text{Coz}(F)\) is a filter.

1. \(\text{czo}(f) = \emptyset\) if and only if \(f = 0\). Hence \(\emptyset = \text{czo}(0) \notin \text{Coz}(F)\), since \(0 \notin F\).

2. Suppose \(\text{czo}(f), \text{czo}(g) \in \text{Coz}(F)\) for some \(f, g \in F\). Since \(F\) is a filter, \(f \wedge g \in F\). Hence, \(\text{czo}(f) \cap \text{czo}(g) = \text{czo}(f \wedge g) \in \text{Coz}(F)\).

3. Suppose \(f \in F\) and \(g \in G^+\) with \(\text{czo}(f) \subseteq \text{czo}(g)\). Note that \(\text{czo}(g) = \text{czo}(f) \cup \text{czo}(g) = \text{czo}(f \vee g)\). Since \(f \in F\) and \(f \leq f \vee g\), it follows that \(f \vee g \in F\), because \(F\) is a filter. Consequently, \(\text{czo}(g) \in \text{Coz}(F)\).

Finally, let \(\text{czo}(g) \notin \text{Coz}(F)\). Then \(g \notin F\). Since \(F\) is an ultrafilter, there exists some \(f \in F\) such that \(g \wedge f = 0\). Therefore, \(\text{czo}(f) \in \text{Coz}(F)\) such that \(\text{czo}(f) \wedge \text{czo}(g) = \emptyset\). Hence, \(\text{Coz}(F)\) is an ultrafilter. \(\square\)

**Lemma 4.6.** Let \(U \in \mathcal{U}(G)\). Then the set \(F(U) = \{f \in G^+ : \text{czo}(f) \in U\}\) is an ultrafilter of \(G^+\).

**Proof.** The proof is similar to the preceding lemma, using the reverse implications of the statements. \(\square\)

We can view the Lemma on Ultrafilters in an alternative manner.
Theorem 4.7. Let \((G, u) \in \mathbf{W}\). For each ultrafilter \(F\) of \(G^+\),
\[ F = F(\text{Coz}(F)) = \{ f \in G^+ : \text{coz}(f) \in \text{Coz}(F)\}. \]
Conversely, for each ultrafilter \(U\) of \(\text{Coz}(G)\), \(U = \text{Coz}(F(U))\). In other
words, \(\Omega: \text{ult}(G^+) \to \mathfrak{U}(G)\) is a bijection.

Proof. It is clear, by the definition of \(\text{Coz}(F)\), that \(F \subseteq \{ f \in G^+ : \text{coz}(f) \in \text{Coz}(F)\}\). For the reverse inclusion, let \(f \in G^+\) be such that \(\text{coz}(f) \in \text{Coz}(F)\). Then there exists \(g \in F\) such that \(\text{coz}(f) = \text{coz}(g)\). If \(f \notin F\), then there exists \(f' \in F\) such that \(f \land f' = 0\), since \(F\) is an ultrafilter. Now
\[
\text{coz}(g \land f') = \text{coz}(g) \cap \text{coz}(f') = \text{coz}(f) \cap \text{coz}(f') = \text{coz}(f \land f') = \emptyset.
\]
Consequently, \(g \land f' \in \bigcap YG = u^\perp = 0 \in F\), which is a contradiction.

The converse can be verified easily. \(\square\)

To summarize, we have the following one-to-one correspondences between ultrafilters and minimal prime ideals of a \(\mathbf{W}\)-object \((G, u)\):
\[
\text{Min}(G) \xleftarrow{\psi} \text{ult}(G^+) \xrightarrow{\Omega} \mathfrak{U}(G)
\]

Endow a topology on \(\mathfrak{U}(G)\) in a similar manner, with the collection of
sets \(\{ F \in \mathfrak{U}(G) : \text{coz}(g) \in F\}\) as a basis for closed sets. It follows that this
space is also compact \(T_1\). We can say more about the spaces \(\mathfrak{U}(G), \text{ult}(G^+),\)
and \(\text{Min}(G)^{-1}\).

Theorem 4.8. All three spaces, \(\mathfrak{U}(G), \text{ult}(G^+),\) and \(\text{Min}(G)^{-1}\) are home-
omorphic to each other.

Proof. We have shown already that \(\text{ult}(G^+\) is homeomorphic to \(\text{Min}(G)^{-1}\).
In a similar fashion, it can be shown that \(\mathfrak{U}(G)\) and \(\text{Min}(G)^{-1}\) are homeo-
morphic to each other. The proof of this is left to the interested reader. \(\square\)

Acknowledgements

The authors gratefully thank their friend and colleague Dr. Warren Wm.
McGovern for his helpful suggestions in the production of this article and
also thank the referees who gave constructive suggestions which helped in
improving the quality of this article.
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