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A note on semi-regular locales

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Abstract. Semi-regular locales are extensions of the classical semi-regular spaces. We investigate the conditions such that semi-regularization is a functor. We also investigate the conditions such that semi-regularization is a reflection or coreflection.

1 Introduction

Recall that a topological space X is said to be a semi-regular space if the family of regular open subsets of X forms a base for X(some authors assume Hausdorffness). The pointless version of semi-regularity is first introduced by J. Paseka and Šmarda in [7], and some properties of semi-regular frames(locales) were studied in [7] and [2]. In this note, we investigate the conditions such that semi-regularization is a functor. We also investigate the conditions such that semi-regularization is a reflection or coreflection. For convenience, we will not distinguish a locale and the corresponding frame, i.e we will use a same letter to represent a locale and its corresponding frame. For more details about locales or frames please refer to Johnstone [4].

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Let X be a locale. An element $x \in X$ is called to be a *regular element* if $\neg\neg x = x$, where $\neg x = \bigvee\{y \in X \mid y \wedge x = 0\}$ be the pseudocomplement of x. The family of all regular elements of X is denoted by $R(X)$. A locale X is called *semi-regular* [7] if the family $R(X)$ of regular elements of X can generates X by joins. It is clear that a topological space X is a semi-regular space if and only if its open sets locale $\mathcal{O}(X)$ is a semi-regular locale.

Many properties of semi-regular topological spaces can be transformed to locales. For example, a regular locale must be semi-regular [7]; the product of a family of semi-regular(strongly Hausdorff semi-regular) locales is semi-regular(strongly Hausdorff semi-regular) [7], etc. Semi-regularity is in general not hereditary even for spatial case. But it is hereditary for dense sublocales [7] and open sublocales as following.

Lemma 1.1. *Let X be a semi-regular(strongly Hausdorff semi-regular) locale and Y an open sublocale. Then Y is semi-regular(strongly Hausdorff semi-regular).*

Proof. First, strongly Hausdorffness is hereditary since for every sublocale inclusion $Y \rightarrowtail X$, the following square is a pullback square.

$$\begin{array}{ccc} Y & \xrightarrow{\triangle} & Y \times Y \\ \downarrow & & \downarrow \\ X & \xrightarrow{\triangle} & X \times X \end{array}$$

If Y is an open sublocale of X, we can regard Y as $\downarrow a$ for some $a \in X$. For $x \in \downarrow a$, write $\neg' x$ for its pseudocomplement in $\downarrow a$, then $\neg' x = \neg x \wedge a$. So for each $y \in \downarrow a$, $y \wedge \neg x = y \wedge \neg' x$ this implies that $\neg'\neg' x = \neg\neg x \wedge a$ for every $x \in \downarrow a$. Thus $R(\downarrow a) = (R(X) \cap \downarrow a) \cup \{a\}$. $\square$

It is well-known known that regularity implies strongly Hausdorffness, and strongly Hausdorffness implies Hausdorffness for spatial locales. I don't know whether strongly Hausdorff semi-regular locales are extensions of Hausdorff semi-regular spaces, i.e. a topological space X is Hausdorff semi-regular if and only if its open sets locale $\mathcal{O}(X)$ is a strongly Hausdorff semi-regular locale.

2 Semi-Regularization

Let X be a locale. We write $SR(X)$ for the locale the corresponding frame of which is the subframe of X generated by $R(X)$, i.e. the frame whose elements have the form $\bigvee S$, $S \subseteq R(X)$. It is clear that $SR(X)$ is a sub-complete lattice of X since every meet of regular elements of X is still a regular element. We write $\neg' a$ for the pseudocomplement of a in $SR(X)$ for $a \in SR(X)$. The following result can be found in [2].

Lemma 2.1. *For each $a \in SR(X)$, $\neg' a = \neg a$.*

By this Lemma, the following result is clear.

Proposition 2.2. *Let X be a locale. Then $SR(X)$ is a semi-regular locale.*

Lemma 2.3. *Let X be a strongly Hausdorff locale and let Y be a sub-complete lattice of X. If $\forall a, b \in X$ with $a \wedge b = 0$, there exist $x, y \in Y$ such that $a \leq x, b \leq y$ and $x \wedge y = 0$, then Y is a strongly Hausdorff locale.*

Proof. Let $\triangle : X \to X \times X$ and $\triangle' : Y \to Y \times Y$ be the diagonals. Then we have $N_0 =\downarrow \{(a, b) \mid a, b \in X, a \wedge b = 0\} =\downarrow \{(x, y) \mid x, y \in Y, x \wedge y = 0\} = N'_0$. Suppose $N'_0 \subseteq J$ be an element of $Y \times Y$. Then it is clear that the lower set $\downarrow J$ in $X \times X$ is an element of $X \times X$ since Y is closed under arbitrary meets, and $N_0 \subseteq\downarrow J$. Write $d' = \bigvee\{x \in Y \mid (x, x) \in J\}$. It suffice to show that $x \wedge y \leq d'$ for $x, y \in Y$ implies that $(x, y) \in J$.

Suppose $x, y \in Y$ and $x \wedge y \leq d'$. Then $(x, y) \in\downarrow J$ since X is strongly Hausdorff. Hence $(x, y) \in J$. $\square$

By Lemma 2.1, we know that $SR(X)$ has the same regular elements as X, i.e. $R(SR(X)) = R(X)$ for each locale X. For $\forall x, y \in X$ with $x \wedge y = 0$, we have $\neg\neg x \wedge \neg\neg y = 0$. Thus the following result is clear by Proposition 2.1.

Corollary 2.4. *Let X be a strongly Hausdorff locale. Then $SR(X)$ is a strongly Hausdorff semi-regular locale.*

We call $SR(X)$ the *semi-regularization* of X. In general, semi-regularization is not a functor even in the spatial case(see [6]). We now consider the situation such that the semi-regularization becomes a functor.

Let $f : X \to Y$ be a locale morphism. We call f a *δ-morphism* if for any regular element $u \in R(Y)$, $f^*(u) \in SR(X)$, i.e. $f^*(u)$ can be represented as a join of regular elements of X. So the statement that $f : X \to Y$ is a δ-morphism is equivalent to saying that we have a commutative square of frame homomorphisms:

$$\begin{array}{ccc} SR(Y) & \longrightarrow & Y \\ {\scriptstyle f^*|_{SR(Y)}}\downarrow & & \downarrow{\scriptstyle f^*} \\ SR(X) & \longrightarrow & X \end{array}$$

Recall that for a continuous map $f : X \to Y$ between two topological spaces X and Y, f is said to be δ-continuous if for each regular open set U in Y, $f^{-1}(U)$ can be represented as a join of regular open sets in X. So our δ-morphisms are just extensions of δ-continuous maps.

Lemma 2.5. *Let $f : X \to Y$ be a locale morphism. The following statements hold:*

1. *If Y is regular then f is a δ-morphism.*
2. *If $f^*(\neg\neg a) = \neg\neg f^*(\neg\neg a)$ for $\forall a \in Y$, then f is a δ-morphism.*
3. *If $f^*(\neg a) = \neg f^*(a)$ for $\forall a \in Y$, then f is a δ-morphism.*

Proof. (1) See [2, Lemma 3.4].

(2) The condition implies that $f^*(u)$ is a regular element of X for every regular element u of Y.

(3) For every regular element $u \in R(Y)$, $f^*(u) = f^*(\neg\neg u) = \neg\neg f^*(u)$ is a regular element of X. $\square$

Morphisms satisfying condition (3) are called *nearly open morphisms* which is equivalent to the condition that for each open sublocale $U \rightarrowtail X$, the image $f(U)$ is dense in some open sublocale of Y (see[1]). We call morphisms satisfying condition (2) *regular open morphisms* which is equivalent to the condition that for each regular element $y \in R(Y)$, $f^*(y)$ is a regular element of X. Clearly,

$$nearly\,open\,morphism \Rightarrow regular\,open\,morphism \Rightarrow \delta - morphism$$

Regular open morphisms, nearly open morphisms and δ-morphisms are respectively stable under composition. The categories of locales with δ-morphisms, regular open morphisms, and nearly open morphisms will be denoted respectively by

$$Loc_\delta,\ Loc_{ro},\ Loc_{no}$$

Write $SRLoc$, $SRLoc_{ro}$, $SRLoc_{no}$ for the subcategories of semi-regular locales with locale morphisms, regular open morphisms, and nearly open morphisms respectively. We have three functors:

$$SR : Loc_\delta \to SRLoc;$$

$$SR : Loc_{ro} \to SRLoc_{ro};$$

$$SR : Loc_{no} \to SRLoc_{no}$$

Now we consider the question of whether we could make the semi-regularization functorial by restricting the objects rather than morphisms.

We will call a locale X *almost regular* if for each regular element $x \in R(X)$, $x = \bigvee\{y \in X \mid y \prec x\}$, where $y \prec x$ means there exists element $z \in X$ such that $z \wedge x = 0$, $z \vee y = 1$. Recall that a topological space X is said to be almost regular if for each regular closed set $A \subseteq X$

and each $x \in X \setminus A$, there exist disjoint open sets U and V of X such that $x \in U$, $A \subseteq V$ (see [8]). So it is clear that almost regular locales are pointless extensions of almost regular spaces.

Lemma 2.6. *X is almost regular if and only if $SR(X)$ is regular.*

Proof. Suppose $x \in R(X)$ and $y \in X$ such that $y \prec x$. Then $\neg\neg y \prec x$ since $\neg\neg\neg y = \neg y$. Hence x can be represented as a join of regular element $\neg\neg y$ with $\neg\neg y \prec x$. This shows that $SR(X)$ is regular. The converse case is clear. □

Corollary 2.7. *X is regular if and only if X is almost regular and semi-regular.*

Proof. The necessity is obvious by Lemma 1.1. Suppose that X is almost regular and semi-regular, then

$x = \bigvee\{a \in R(X) \mid a \leq x\} = \bigvee\{\bigvee\{y \in X \mid y \prec a\} \mid a \in R(X), a \leq x\} = \bigvee\{y \in X \mid y \prec x\}$

for each $x \in X$. □

Similar to the proof of Lemma 2.3 (1), we can show the following result.

Lemma 2.8. *Let $f : X \to Y$ be a locale morphism and Y be almost regular, then f is a δ-morphism.*

We write *ARLoc* and *RLoc* for the categories of almost regular locales with locale morphisms and the category of regular locales with locale morphisms respectively.

Theorem 2.9. *SR : ARLoc → RLoc is a functor which is left adjoint to the inclusion RLoc → ARLoc, i.e. $SR(X)$ is a regular reflection for every almost regular locale X.*

Let L be a locale. We write IdL for the frame of all ideas of L. For each $I \in IdL$, it is clear that $\neg I = \downarrow \neg \bigvee I$. Thus $I \in R(IdL)$ if and only if I be a principle ideal $\downarrow a$ with $a \in R(L)$.

Lemma 2.10. *Let L be a locale. The following conditions are equivalent:*

1. *$SR(IdL)$ is isomorphic to $IdR(L)$;*
2. *$\neg(a \wedge b) = \neg a \vee \neg b$ for $\forall a, b \in L$;*
3. *Each $a \in R(L)$ has a complement in L.*

Proof. (1)⇒(3) Let $\hat{R}(L)$ be the sublattice of L generated by $R(L)$, then $SR(IdL) \cong Id\hat{R}(L)$. Hence condition (1) implies that $\hat{R}(L) \cong R(L)$, i.e. $\hat{R}(L)$ is a boolean algebra. Thus each $a \in R(L)$ has a complement in L.

(3) ⇒ (2) Clear.

(2) ⇒ (3) Suppose $a \in R(L)$, then $a \vee \neg a = \neg\neg a \vee \neg a = \neg(\neg a \wedge a) = 1$, which implies that $\neg a$ is a complement of a in L.

(3) ⇒(1) If each $a \in R(L)$ has a complement in L, then $\hat{R}(L) \cong R(L)$. Thus $SR(IdL) \cong IdR(L)$. □

A locale satisfying the above equivalent conditions will be called a *b-locale*. For every locale L, we have a canonical map $\bigvee : SR(IdL) \to L$ which is clearly a frame homomorphism.

Theorem 2.11. *Let L be a b-locale. Then the canonical morphism $L \to SR(IdL)$ becomes a compact zero-dimensional reflection of L, and it is an embedding if and only if L is semi-regular.*

We call a locale X *almost compact* if every $S \subseteq R(X)$, $\bigvee S = 1$ implies that there exists a finite set $F \subseteq S$ such that $\bigvee F = 1$.

Corollary 2.12. *Let L be a almost compact b-locale. Then the canonical map $\bigvee : SR(IdL) \to SR(L)$ is an isomorphism.*

3 Semi-Regularization as Reflection and Coreflection

For every locale X, $SR : X \to SR(X)$ is clearly nearly open, thus a regular open morphism and a δ-morphism.

Theorem 3.1. *The functors $SR : Loc_\delta \to SRLoc$, $SR : Loc_{ro} \to SRLoc_{ro}$, $SR : Loc_{no} \to SRLoc_{no}$ are left adjoint to the inclusions $SRLoc \rightarrowtail Loc_\delta$, $SRLoc_{ro} \rightarrowtail Loc_{ro}$, $SRLoc_{no} \rightarrowtail Loc_{no}$ respectively, i.e. the categories $SRLoc$, $SRLoc_{ro}$ and $SRLoc_{no}$ are respectively reflective in the categories Loc_δ, Loc_{ro} and Loc_{no}.*

Now we consider the question of whether the semi-regularization could be made to be a coreflection, i.e. SR becomes a right adjoint to the inclusion $SRLoc \to Loc$.

Let X be a locale. We know that $SR(X)$ is closed under arbitrary joins and meets in X, hence the frame inclusion $SR(X) \rightarrowtail X$ has a left adjoint $X \to SR(X)$ which assigns to every element x of X the least element $\bar{x} \in SR(X)$ with $x \leq \bar{x}$.

Definition 3.2. A locale X is said to be a *c-locale* if for every $x, y \in X$, $\overline{x \wedge y} = \bar{x} \wedge \bar{y}$.

Since the left adjoint $X \to SR(X)$ of the frame inclusion $SR(X) \rightarrowtail X$ always preserves joins, so a locale X is a c-locale if and only if the left adjoint $X \to SR(X)$ is a frame morphism.

Every semiregular locale is a c-locale.

Example 3.3. Let X be a locale such that its least element 0 is a prime element (for example unit interval $[0, 1]$ regarded as a locale; the open sets lattice of a infinite set with the topology of complements of finite sets). Then $SR(X)$ is the two points lattice $\{0, 1\}$. For each $x \in X$,

$$\bar{x} = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

hence $\overline{x \wedge y} = \bar{x} \wedge \bar{y}$. Thus X is a c-locale.

If X is a c-locale, then for each $a \in X$ and $b \in SR(X)$, $\overline{a \wedge b} = \bar{a} \wedge b$ implies that the frame inclusion $SR(X) \to X$ is an open frame homomorphism.

We call a locale morphism $f : X \to Y$ c-locale morphism if for each $y \in Y$, $\overline{f^*(\bar{y})} = \overline{f^*(y)}$. Every locale morphism between semiregular locales is a c-locale morphism.

If $f : X \to Y$ and $g : Y \to Z$ are both c-locale morphisms, then $\overline{f^*g^*(z)} = \overline{f^*\overline{g^*(z)}} = \overline{f^*\overline{g^*(\bar{z})}} = \overline{f^*\overline{g^*(z)}} = \overline{f^*g^*(z)}$. It implies that the composition gf is a c-locale morphism. Thus we have a category $CLoc$ of c-locales with c-locale morphisms.

Let X and Y be c-locales and $f : X \to Y$ be a c-locale morphism. Define $SR(f) : SR(X) \to SR(Y)$ by

$$SR(f)^*(y) = \overline{f^*(y)}$$

$SR(f)^*$ clearly preserves arbitrary joins. For $\forall x, y \in SR(Y), SR(f)^*(x \wedge y) = \overline{f^*(x \wedge y)} = \overline{f^*(x) \wedge f^*(y)} = \overline{f^*(x)} \wedge \overline{f^*(y)} = SR(f)^*(x) \wedge SR(f)^*(y)$, hence $SR(f)^*$ is a frame homomorphism.

Theorem 3.4. *$SR : CLoc \to SRLoc$ is a functor which is right adjoint to the inclusion $SRLoc \to CLoc$, i.e. $SR(X)$ is the semi-regular coreflection of X for every c-locale X.*

Proof. Let X and Y be both c-locales and $f : X \to Y$ a c-locale morphism. We have a commutative square:

$$\begin{array}{ccc} SR(X) & \longrightarrow & X \\ {\scriptstyle SR(f)}\downarrow & & \downarrow{\scriptstyle f} \\ SR(Y) & \longrightarrow & Y \end{array}$$

So if Y is semi-regular, then $SR(Y) \to Y$ is an isomorphism and f has a unique factorization through the morphism $SR(X) \to X$. □

A locale morphism $f : X \to Y$ is called a *$c\delta$-morphism* if it is simultaneously a c-morphism and a δ-morphism. We write $CLoc_\delta$ for the category of c-locales with $c\delta$-morphisms.

Corollary 3.5. *The category $SRLoc$ of semi-regular locales is simultaneously reflective and coreflective in the category $CLoc_\delta$ of c-locales and $c\delta$-morphisms.*

4 Semi-Regularization as Booleanization

In this section we consider the situation where semi-regularization is a Boolean algebra.

Lemma 4.1. *Let X be a locale. The following conditions are equivalent:*

1. *$SR(X)$ is a Boolean algebra.*
2. *$R(X)$ is closed under arbitrary joins.*
3. *$\neg\neg \bigvee S = \bigvee\{\neg\neg s \mid s \in S\}$ for $\forall S \subseteq X$.*

Proof. (1)⇒(2) The statement that $SR(X)$ is a Boolean algebra implies that $R(X)$ is a Boolean algebra and closed under arbitrary joins.

(2)⇒(3) Suppose that $R(X)$ is closed under arbitrary joins, then for every $S \subseteq X$, $\bigvee\{\neg\neg s \mid s \in S\} = \neg\neg \bigvee\{\neg\neg s \mid s \in S\} \geqslant \neg\neg \bigvee S$. Thus $\neg\neg \bigvee S = \bigvee\{\neg\neg s \mid s \in S\}$.

(3)⇒(1) It suffices to show that every regular element is complemented and $R(X)$ is closed under arbitrary joins. Suppose $a \in R(X)$, then $a \vee \neg a = \neg\neg a \vee \neg a = \neg(\neg a \wedge a) = 1$. Thus a is complemented. For every $S \subseteq R(X)$, $\bigvee S = \neg\neg \bigvee S$, hence $R(X)$ is closed under arbitrary joins. □

We will call a frame satisfying the above equivalent conditions *cb-frame*. The category of cb-frames and frame homomorphisms is denoted by $CBFrm$.

Theorem 4.2. *The corresponding of Booleanization* $CBFrm \to Bool$ *from the category of cb-frames to the category of complete Boolean algebras is a functor which is equivalent to the semi-regularization functor* $SR : CBFrm \to SRFrm$.

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