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Countable composition closedness and integer-valued continuous functions in pointfree topology

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Abstract. For any archimedean f-ring A with unit in which $a \wedge (1-a) \leq 0$ for all $a \in A$, the following are shown to be equivalent:
1. A is isomorphic to the l-ring $\mathfrak{Z}L$ of all integer-valued continuous functions on some frame L.
2. A is a homomorphic image of the l-ring $C_{\mathbb{Z}}(X)$ of all integer-valued continuous functions, in the usual sense, on some topological space X.
3. For any family $(a_n)_{n\in\omega}$ in A there exists an l-ring homomorphism $\varphi : C_{\mathbb{Z}}(\mathbb{Z}^\omega) \to A$ such that $\varphi(p_n) = a_n$ for the product projections $p_n : \mathbb{Z}^\omega \to \mathbb{Z}$.

This provides an integer-valued counterpart to a familiar result concerning real-valued continuous functions.

The fundamental fact that the lattice-ordered rings $\mathfrak{R}L$ of real-valued continuous functions on frames L are characterized as the countable composition closed (c^3) archimedean f-rings with unit, first stated by Isbell [4] and rather later given a somewhat more detailed proof by Madden and Vermeer [7], raised the obvious question whether the integer-valued counterparts $\mathfrak{Z}L$ of the $\mathfrak{R}L$ have an analogous characterization. That did,

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indeed, turn out to be the case in due course, but the approach of [4] and [7] could hardly be used for this purpose. While the original arguments regarding $\mathfrak{R}L$ were firmly grounded in the setting of the Yosida representation of archimedean f-rings, the treatment of $\mathfrak{Z}L$ in Banaschewski [1] took place in the context that appeared most natural for such questions at that later stage: the functoriality of the correspondence $L \longmapsto \mathfrak{Z}L$. This paper presents a detailed account of this (which has never been published except for some further talks over the years), prompted by the fact that a new version of the proof has recently evolved which is somewhat simpler and rather more appealing than the original one. In addition, it will be shown that a simple modification of the arguments concerning $\mathfrak{Z}L$ will provide a proof for the corresponding result for the $\mathfrak{R}L$.

We begin with a brief account of the background involved here. For general aspects of frames we refer to Picado and Pultr [8] and for details concerning the function ring functors to Banaschewski [2].

Recall that the integer-valued continuous functions on a frame L may be viewed as the frame homomorphisms to L from the frame $\mathfrak{O}\mathbb{Z}$ of open sets of the discrete space $\mathbb{Z}$ of integers (conveniently described by the maps $\alpha : \mathbb{Z} \to L$ such that $\alpha(k) \wedge \alpha(l) = 0$ for $k \neq l$ and $\bigvee\{\alpha(m)|m \in \mathbb{Z}\} = e$, the unit of L), and $\mathfrak{Z}L$ is then the l-ring whose elements are the homomorphisms $\mathfrak{O}\mathbb{Z} \to L$, with its operations derived from those of $\mathbb{Z}$ as l-ring in the familiar way. Any $\mathfrak{Z}L$ is an archimedean f-ring with unit $\mathbf{1}$ such that

$$\alpha \wedge (\mathbf{1} - \alpha) \leq \mathbf{0} \text{ for all } \alpha \in \mathfrak{Z}L,$$

where $\mathbf{1}$ and $\mathbf{0}$ are the constant functions determined by 1 and 0 of $\mathbb{Z}$. We refer to the f-rings with unit which satisfy this condition as $\mathbb{Z}$-rings.

Now, any $\mathfrak{Z}L$ is actually isomorphic to $\mathfrak{Z}M$ for the subframe M of L generated by the complemented elements of L so that we may take all frames considered here as 0-dimensional without loss of generality.

Concerning the relation to the classical function rings, note that $\mathfrak{Z}(\mathfrak{O}X) \cong C_{\mathbb{Z}}(X)$ where the latter is the l-ring of all integer-valued continuous functions, in the usual sense, on the topological space X and $\mathfrak{O}X$ the frame of open sets of X.

Next, any frame homomorphism $h : L \to M$ clearly determines a map $\mathfrak{Z}h : \mathfrak{Z}L \to \mathfrak{Z}M$, $\gamma \longmapsto h\gamma$, which is in fact an l-ring homomorphism, resulting in the functor $\mathfrak{Z}$ from the category $\mathbb{O}\mathbb{D}$**Frm** of 0-dimensional frames to the category $\mathbb{A}\mathbb{Z}$ of archimedean $\mathbb{Z}$-rings. Further, $\mathfrak{Z}$ has a left adjoint $\mathfrak{K} : \mathbb{A}\mathbb{Z} \to \mathbb{O}\mathbb{D}\mathbf{Frm}$, taking each $A \in \mathbb{A}\mathbb{Z}$ to the frame $\mathfrak{K}A$ of its archimedean kernels, that is, the l-ring ideals J of A for which A/J is archimedean, and each $\varphi : A \to B$ in $\mathbb{A}\mathbb{Z}$ to the frame homomorphism $\mathfrak{K}\varphi : \mathfrak{K}A \to \mathfrak{K}B$, sending $J \in \mathfrak{K}A$ to the archimedean kernel of B generated by $\varphi[J]$. Further, the adjunction maps are

$$\zeta_A : A \to \mathfrak{Z}\mathfrak{K}A,\ a \mapsto \hat{a},\ \hat{a}(\{m\}) = <(1 - |a - m|^+)>,$$

$< \cdot >$ for the archimedean kernel generated by $\cdot$, and

$$\delta_L : \mathfrak{K}\mathfrak{Z}L \to L,\ J \mapsto \bigvee\{coz(\gamma)|\gamma \in J\},$$

where $coz(\gamma) = \gamma(\mathbb{Z} - \{0\})$. Now, as is familiar, $\mathfrak{Z}$ and $\mathfrak{K}$ induce an adjoint equivalence between the subcategories of $\mathbb{A}\mathbb{Z}$ and $\mathbb{O}\mathbb{D}$**Frm** determined by the conditions ζ_A is an isomorphism and δ_L is an isomorphism, respectively. The latter consists of all $\mathbb{Z}$-*complete* frames, meaning the 0-dimensional frames which are complete for the uniformity given by their countable covers of pairwise disjoint elements (or, equivalently, the 0-dimensional Lindelöf frames, provided the Axiom of Countable Choice is assumed). Note that, for such L,

$$\delta_L^{-1}(s) = \{\gamma \in \mathfrak{Z}L | coz(\gamma) \leq s\}$$

by the general properties of the coz-map. On the other hand, to provide a description of the analogous subcategory of $\mathbb{A}\mathbb{Z}$ which does not depend on the adjunction map ζ_A is exactly the purpose of this note.

The condition involved here is as follows:

($\mathbb{Z}c^3$) *For any family $(a_n)_{n\in\omega}$ in A, there exists an l-ring homomorphism $\varphi : C_{\mathbb{Z}}(\mathbb{Z}^\omega) \to A$ such that $\varphi(p_n) = a_n$ for the product projection $p_n : \mathbb{Z}^\omega \to \mathbb{Z}$.*

We call the $A \in \mathbb{A}\mathbb{Z}$ for which this holds *countable $\mathbb{Z}$-composition closed* ($\mathbb{Z}c^3$ for short), and note that the $C_{\mathbb{Z}}(X)$ are obviously of this type: any $(f_n)_{n\in\omega}$ in $C_{\mathbb{Z}}(X)$ determines $f : X \to \mathbb{Z}^\omega$ such that $p_n f = a_n$

by the nature of the products of spaces, and the corresponding l-ring homomorphism

$$C(f) : C_{\mathbb{Z}}(\mathbb{Z}^{\omega}) \to C_{\mathbb{Z}}(X), \quad g \longmapsto gf,$$

trivially has the required property.

Obviously, $(\mathbb{Z}c^3)$ is the integer-valued version of the familiar condition (c^3) which has $\varphi : C(\mathbb{R}^{\omega}) \to A$ in place of $\varphi : C_{\mathbb{Z}}(\mathbb{Z}^{\omega}) \to A$. The corresponding archimedean f-rings A with unit, called countable composition closed, are then exactly the A isomorphic to some $\mathfrak{R}L$, the l-ring of all *real-valued* continuous functions on some frame L, as mentioned earlier. In this context it is worth pointing out that the original definition of (c^3) was formulated in terms of the Yosida representation while the version just described was introduced by Madden only some 15 years ago. It was undoubtedly a crucial contribution to the subject to place this condition entirely within the setting of the function ring functor involved without which the present treatment of $\mathfrak{Z}L$ would not have been possible.

We first give an account of some basic facts concerning the copowers of $\mathfrak{O}\mathbb{Z}$ which will be needed later on. For this, let S be any set and

$$k_S : \bigoplus_S \mathfrak{O}\mathbb{Z} \to \mathfrak{O}(\mathbb{Z}^S)$$

such that $k_S j_t = \mathfrak{O}p_t$ for each $t \in S$, where $j_t : \mathfrak{O}\mathbb{Z} \to \bigoplus_S \mathfrak{O}\mathbb{Z}$ is the coproduct injection and $p_t : \mathbb{Z}^S \to \mathbb{Z}$ the product projection, and put

$$\tau_S = \mathfrak{Z}k_S : \mathfrak{Z}(\bigoplus_S \mathfrak{O}\mathbb{Z}) \to \mathfrak{Z}\mathfrak{O}(\mathbb{Z}^S).$$

Note that, by a standard computation, k_S is the reflection map to spatial frames; further, since $\mathfrak{Z}\mathfrak{O}X \cong C_{\mathbb{Z}}(X)$, as noted earlier, $\mathfrak{Z}(\bigoplus_S \mathfrak{O}\mathbb{Z}) \cong C_{\mathbb{Z}}(\mathbb{Z}^S)$.

Now we have the following, not entirely unknown but put here for the reader's convenience.

Lemma 1.
(1) *Any* $\bigoplus_S \mathfrak{O}\mathbb{Z}$ *is* $\mathbb{Z}$*-complete.*
(2) *For* (*at most*) *countable* S, k_S *is an isomorphism.*
(3) *For any* S, τ_S *is an isomorphism.*
(4) *Any* $\gamma : \mathfrak{O}\mathbb{Z} \to \bigoplus_S \mathfrak{O}\mathbb{Z}$ *factors through some partial coproduct* $\bigoplus_T \mathfrak{O}\mathbb{Z}$ *with countable* $T \subseteq S$.

Proof.

1. Since $\mathfrak{O}\mathbb{Z}$ is complete in its uniformity of all covers, $\bigoplus_S \mathfrak{O}\mathbb{Z}$ is complete in the corresponding coproduct uniformity, and since this is coarser than its $\mathbb{Z}$-uniformity, $\bigoplus_S \mathfrak{O}\mathbb{Z}$ is also complete in the latter.

2. As noted earlier, k_S is the reflection map to spatial frames. On the other hand, the coproduct uniformity of $\bigoplus_S \mathfrak{O}\mathbb{Z}$ has a countable basis, as S is countable, and since $\bigoplus_S \mathfrak{O}\mathbb{Z}$ is complete in this, as already mentioned, it is spatial by a result if Isbell [4].

3. $\mathfrak{Z}k_S$ is one-one because k_S is dense since

$$\emptyset = k_S(j_{t_1}(U_1) \wedge \cdots \wedge j_{t_n}(U_n)) = p_{t_1}^{-1}[U_1] \cap \cdots \cap p_{t_n}^{-1}[U_n]$$

implies that some $U_k = \emptyset$ and hence $j_{t_1}(U_1) \wedge \cdots \wedge j_{t_n}(U_n) = 0$ trivially. To see $\mathfrak{Z}k_S$ is onto consider the diagram

$$\begin{tikzcd}
\bigoplus_S \mathfrak{O}\mathbb{Z} \arrow[rr, "k_S"] & & \mathfrak{O}(\mathbb{Z}^S) \\
& \mathfrak{O}(\mathbb{Z}^T) \arrow[ur, "\mathfrak{O}p"] & \\
\bigoplus_T \mathfrak{O}\mathbb{Z} \arrow[uu, "j"] \arrow[ur, "k_T"] & & \mathfrak{O}\mathbb{Z} \arrow[ul, "\psi"] \arrow[uu, "\varphi"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{Z}^S \arrow[dr, "p"] \arrow[dd, "f"'] & \\
& \mathbb{Z}^T \arrow[dl, "g"] \\
\mathbb{Z} &
\end{tikzcd}$$

where φ is arbitrary, f corresponds to φ, $T \subseteq S$ is countable such that f factors through the corresponding partial product projection p (as provided by a familiar classical result), $\psi = \mathfrak{O}g$ so that $\varphi = (\mathfrak{O}p)\psi$, and j is the partial coproduct injection. Now for the coproduct maps $i_t : \mathfrak{O}\mathbb{Z} \to \bigoplus_T \mathfrak{O}\mathbb{Z}$ and the product maps $q_t : \mathbb{Z}^T \to \mathbb{Z}$,

$$(\mathfrak{O}p)k_T i_t = (\mathfrak{O}p)(\mathfrak{O}q_t) = \mathfrak{O}(q_t p) = \mathfrak{O}p_t = k_S j_t = (k_S j)i_t$$

hence $(\mathfrak{O}p)k_T = k_S j$ so that $\mathfrak{O}p = k_S j k_T^{-1}$ by (2) and therefore

$$k_S(jk_T^{-1}\psi) = (\mathfrak{O}p)(\mathfrak{O}g) = \mathfrak{O}f = \varphi$$

showing $\mathfrak{Z}k_S$ is onto.

4. By (1), $\bigoplus_S \mathfrak{O}\mathbb{Z}$ is Lindelöf, as noted earlier, so that each $\gamma(\{m\})$, being complemented, is a Lindelöf element of $\bigoplus_S \mathfrak{O}\mathbb{Z}$ and hence a join of countably many elements

$$j_{t_1}(U_1) \wedge \cdots \wedge j_{t_n}(U_n).$$

Consequently, there exists a countable $T_m \subseteq S$ such that $\gamma(\{m\})$ belongs to the image in $\bigoplus_S \mathfrak{O}\mathbb{Z}$ of the corresponding partial coproduct, and the countable $T = \bigcup\{T_m | m \in \mathbb{Z}\}$ then has the stated property.

$\square$

Remark 1. Note that, in view of (2) above, $(\mathbb{Z}c^3)$ is equivalent to the condition that, for any countable $S \subseteq A$, there exists a homomorphism $\varphi : \mathfrak{Z}(\bigoplus_S \mathfrak{O}\mathbb{Z}) \to A$ such that $\varphi(j_a) = a$ for the coproduct injections j_a. It will be convenient later on to use this form of the condition.

Lemma 2. *Any archimedean image of a $\mathfrak{Z}L$ is isomorphic to a $\mathfrak{Z}M$.*

Proof. Let A be of the stated kind and $\varphi : \mathfrak{Z}L \to A$ a corresponding onto homomorphism. Now, note first that L may be taken as $\mathbb{Z}$-*complete* because $\mathfrak{Z}\delta_L : \mathfrak{Z}\mathfrak{K}\mathfrak{Z}L \to \mathfrak{Z}L$ is an isomorphism by the properties of the adjunction between $\mathfrak{Z}$ and $\mathfrak{K}$, and the same holds for any $\delta_{\mathfrak{K}A} : \mathfrak{K}\mathfrak{Z}\mathfrak{K}A \to \mathfrak{K}A$ which in turn makes $\mathfrak{K}A$ $\mathbb{Z}$-complete. Hence, by the earlier description of $(\delta_L)^{-1}$,

$$Ker(\varphi) = \{\gamma \in \mathfrak{Z}L | coz(\gamma) \leq s\}, \quad s = \delta_L(Ker\varphi),$$

so that, for $\nu_s = (\cdot) \vee s : L \to \uparrow s = \{x \in L | x \geq s\}$,

$$\begin{aligned} coz(\gamma) \leq s \text{ iff } s = \nu_s(coz(\gamma)) = coz(\nu_s\gamma) \\ \text{iff } \nu_s\gamma = \mathbf{0} \text{ in } \mathfrak{Z}(\uparrow s) \text{ iff } (\mathfrak{Z}\nu_s)(\gamma) = \mathbf{0}, \end{aligned}$$

the second step because s is the zero of $\uparrow s$, and thus $Ker(\varphi) = Ker(\mathfrak{Z}\nu_s)$.

Further, $\mathfrak{Z}\nu_s : \mathfrak{Z}L \to \mathfrak{Z}(\uparrow s)$ is onto, which evidently amounts to saying that any countable cover P of $\uparrow s$ by pairwise disjoint elements is the image by ν_s of such a cover Q of L. Now P is of course a cover of L and L is 0-dimensional Lindelöf so that the cover of L by all the complemented elements below some $u \in P$ has a countable subcover $C = \{a_n | n \in \omega\}$ which then determines $Q = \{b_n | n \in \omega\}$ where

$$b_0 = a_0, \ b_{n+1} = (a_1 \vee \cdots \vee a_n)^* \wedge a_{n+1},$$

(* for pseudocomplement which means complement for these elements) and one readily sees that the b_n are pairwise disjoint such that $a_0 \vee a_1 \vee \cdots \vee a_n = b_0 \vee b_1 \vee \cdots \vee b_n$. On the other hand, for each n, $\nu_s(b_n) \leq u$ for some $u \in P$ by the definition of the b_n which implies $\nu_s[Q] = P$, as desired, and it then follows that

$$A \cong \mathfrak{Z}L/Ker(\varphi) \cong \mathfrak{Z}L/Ker(\mathfrak{Z}\nu_s) \cong \mathfrak{Z}(\uparrow s).$$

$\square$

Proposition. *The following are equivalent for any archimedean $\mathbb{Z}$-ring A:*

(1) *A is isomorphic to some $\mathfrak{Z}L$.*
(2) *A is a homomorphic image of some $C_{\mathbb{Z}}(X)$.*
(3) *A is $\mathbb{Z}c^3$.*

Proof. (1) $\Rightarrow$ (2). We show any $\mathfrak{Z}L$ is of the stated kind. For this, take $h : \bigoplus_{\mathfrak{Z}L} \mathfrak{O}\mathbb{Z} \to L$ such that $hj_\alpha = \alpha$ for the coproduct injections. Then $\mathfrak{Z}h : \mathfrak{Z}(\bigoplus_{\mathfrak{Z}L} \mathfrak{O}\mathbb{Z}) \to \mathfrak{Z}L$ is trivially onto, and by Lemma 1 this provides an onto homomorphism $C_{\mathbb{Z}}(\mathbb{Z}^{\mathfrak{Z}L}) \to \mathfrak{Z}L$.

(2) $\Rightarrow$ (3). As already noted, any $C_{\mathbb{Z}}(X)$ is $\mathbb{Z}c^3$, and the same then obviously holds for any homomorphic image of any $C_{\mathbb{Z}}(X)$.

(3) $\Rightarrow$ (1) For any A of the stated kind, take any $\mathfrak{Z}L$ containing A as a sub-l-ring (as provided, say, by the adjunction map $\zeta_A : A \to \mathfrak{Z}\mathfrak{K}A$) and consider $h : \bigoplus_A \mathfrak{O}\mathbb{Z} \to L$ such that $hj_\alpha = \alpha$ for the coproduct injections j_α. We show that $Im(\mathfrak{Z}h) = A$ which proves (1) in view of Lemma 2.

Since it is obvious from the definition of h that $A \subseteq Im(\mathfrak{Z}h)$ it only remains to show $Im(\mathfrak{Z}h) \subseteq A$. Now, for any $\gamma : \mathfrak{O}\mathbb{Z} \to \bigoplus_A \mathfrak{O}\mathbb{Z}$, Lemma

1 supplies a countable $S \subseteq A$ and $\tilde{\gamma} : \mathfrak{O}\mathbb{Z} \to \bigoplus_S \mathfrak{O}\mathbb{Z}$ such that $\gamma = j\tilde{\gamma}$ for the partial coproduct injection $j : \bigoplus_S \mathfrak{O}\mathbb{Z} \to \bigoplus_A \mathfrak{O}\mathbb{Z}$. On the other hand, A being $\mathbb{Z}c^3$, we have a homomorphism $\varphi : \mathfrak{Z}(\bigoplus_S \mathfrak{O}X) \to A$ such that $\varphi(i_\alpha) = \alpha$ for the coproduct injections i_α by Remark 1. Further, the $\mathbb{Z}$-completeness of $\bigoplus_S \mathfrak{O}\mathbb{Z}$ (Lemma 1) supplies $k : \bigoplus_S \mathfrak{O}X \to L$ such that $\mathfrak{Z}k = l\varphi$ where $l : A \to \mathfrak{Z}L$ is the identical embedding (Banaschewski [2]), and then $k = hj$ because

$$ki_\alpha = l\varphi(i_\alpha) = \alpha = hj_\alpha = (hj)i_\alpha$$

for all $\alpha \in S$. Consequently,

$$(\mathfrak{Z}h)(\gamma) = h\gamma = hj\tilde{\gamma} = k\tilde{\gamma} = \mathfrak{Z}k(\tilde{\gamma}) = l\varphi(\tilde{\gamma}) \in A,$$

showing $Im(\mathfrak{Z}h) \subseteq A$. □

Remark 2. The original proof of the Proposition in Banaschewski [1] was rather more complicated than the present one, which only recently evolved. The crucial new step is Lemma 2 which was motivated by Banaschewski, Bhattacharjee, and Walters-Wayland [3]. What seems interesting here is that this is absolutely trivial once one has the $\mathbb{Z}c^3$ characterization of the $\mathfrak{Z}L$ but can readily be proved without that and then be utilized with advantage in proving that characterization.

Remark 3. It should be added here that almost the same arguments used above can be applied to the l-ring $\mathfrak{R}L$ of real-valued continuous functions instead of $\mathfrak{Z}L$, to provide a proof of the analogous proposition:

The following are equivalent for any archimedean f-ring A with unit:
(1) *A is isomorphic to some $\mathfrak{R}L$.*
(2) *A is a homomorphic image of some $C(X)$.*
(3) *A is c^3.*

What has to be done here is obvious: in the given text, turn all the statements about $\mathfrak{O}\mathbb{Z}$ and $\mathfrak{Z}L$ into statements about the frame $\mathcal{L}(\mathbb{R})$ of reals and the l-ring $\mathfrak{R}L$ of real-valued continuous functions, respectively, and replace 0-dimensionality by complete regularity and $\mathbb{Z}$-completeness by its obvious counterpart realcompleteness (= completeness relative to the uniformity defined by the images of the standard uniform covers of $\mathcal{L}(\mathbb{R})$ by the $\gamma \in \mathfrak{R}L$). With this modification, the proof of Lemma 1, (1) - (3), and of the Proposition turn into proofs of their modified versions. In the case of Lemma 1(4) one uses the familiar fact that, for any $\gamma : \mathcal{L}(\mathbb{R}) \to \bigoplus_S \mathcal{L}(\mathbb{R})$, the elements $\gamma(p,q),\ p,q \in \mathbb{Q}$, are Lindelöf because $\bigoplus_S \mathcal{L}(\mathbb{R})$ is Lindelöf. Finally regarding Lemma 2, the argument for $Ker(\varphi) = Ker(\mathfrak{R}\nu_S)$, now for $\varphi : \mathfrak{R}L \to A$, is certainly valid again, but that the modified $\mathfrak{R}\nu_S : \mathfrak{R}L \to \mathfrak{R}(\uparrow s)$ is onto very obviously requires a new argument - fortunately readily available. Note that, in analogy with the case of $\mathfrak{Z}L$, L here may be taken as realcomplete which makes it Lindelöf (Banaschewski [2]) and therefore *normal* so that $\mathfrak{R}\nu_S$ is onto by the pointfree Tietze Theorem (Li and Wang [6]).

Remark 4. Two further modifications of the Proposition can be proved in a similar way, one concerning the archimedean l-groups with specified weak order unit and the other where the order unit is assumed to be singular as well, which corresponds to the $\mathfrak{R}L$-case and the $\mathfrak{Z}L$-case, respectively. We omit the details.

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