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On MV-algebras of non-linear functions

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Dedicated to Bernhard Banaschewski on the Occasion of his 90th Birthday

Abstract. In this paper, the main results are: a study of the finitely generated MV-algebras of continuous functions from the n-th power of the unit real interval I to I; a study of Hopfian MV-algebras; and a category-theoretic study of the map sending an MV-algebra as above to the range of its generators (up to a suitable form of homeomorphism).

1 Introduction

MV-algebras are the structures corresponding to Łukasiewicz many valued logic, in the same sense in which Boolean algebras correspond to classical logic. Usually MV-algebras (in particular the finitely presented ones) are represented by McNaughton functions, which are continuous piecewise linear functions, but it could be interesting to represent MV-algebras with non-linear functions, especially for applications. One could relax the linearity requirement and consider piecewise polynomial functions, which are important for several reasons, for instance they are the subject of the cel-

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ebrated Pierce-Birkhoff conjecture, see [2], and include, in one variable, the spline functions, a kind of functions which have been deeply studied, see [14] and [15]. Other examples are Lyapunov functions used in the study of dynamical systems, see [8], and logistic functions, see [17].

We stick to continuous functions, despite that for certain applications it could be reasonable to use discontinuous functions, for instance in order to model arbitrary signals in Fourier analysis. Continuous functions are preferable for technical reasons; for instance, they preserve compact sets. So, our MV-algebras of interest will be the MV-algebras of all continuous functions from $[0,1]^n$ to $[0,1]$, which we will denote by C_n. We call also M_n the MV-algebra of McNaughton functions from $[0,1]^n$ to $[0,1]$, that is, M_n is the set of all continuous piecewise affine functions with integer coefficients. M_n is isomorphic to the free MV-algebra in n generators. Then the free MV-algebras (over n generators) coincide with the isomorphic copies of M_n. In this paper, as a rule, we prefer not to identify isomorphic MV-algebras of functions, because they can consist of functions with very diverse geometric properties, which may be relevant for applications.

The main results of this paper are:

1. A study of the finitely generated MV-subalgebras of C_n (Theorem 2.8).

2. A study of Hopfian MV-algebras (Section 3).

3. A category-theoretic study of the map sending a finitely generated MV-subalgebra of C_n to the range of its generators, modulo definable homeomorphisms (Theorem 4.6).

2 MV-algebras of non-linear functions

For definitions and preliminaries we refer to the standard references [4, 11].

Let $H = (h_1, \ldots, h_n)$ be an n-tuple of functions from $[0,1]^n$ to $[0,1]$. The corresponding set is $\sim H = \{h_1, \ldots, h_n\}$. Assume that $(h_1, \ldots, h_n)$ is a homeomorphism from $[0,1]^n$ to $[0,1]^n$. Then the map a_H from C_n to C_n such that $a_H(g) = g(h_1, \ldots, h_n)$ is an MV-algebra automorphism of C_n, and it sends the i-th projection π_i to h_i, for every $i = 1, \ldots, n$. Since $\pi_1, \ldots, \pi_n$ generate M_n, it follows that $h_1, \ldots, h_n$ generate a copy of M_n.

Note that the MV-algebra generated by $\sim H$ consists of the functions of the form $f \circ H$ with $f \in M_n$, that is, the piecewise linear functions in the variables $h_1(x_1, \dots, x_n), \dots, h_n(x_1, \dots, x_n)$. So these functions are piecewise linear up to a change of coordinates.

Since there are continuum many homeomorphisms from $[0,1]^n$ to $[0,1]^n$ whereas each copy of M_n is countable, we obtain:

Proposition 2.1. *C_n contains continuum many copies of M_n.*

Definition 2.2. (see [9]) Let $C \subseteq [0,1]^n$ and $D \subseteq [0,1]^m$. We call *definable map* from C to D any m-tuple of McNaughton functions in M_n which sends C to D. We call *definable homeomorphism* between C and D an invertible definable map from C to D whose inverse is a definable map from D to C.

Given a subset C of $[0,1]^n$, it is useful to denote by $M_n|_C$ the MV-algebra of McNaughton functions restricted to C.

In the rest of this paper we will treat several kinds of subsets of the n-cube. To this aim we find it useful to introduce the following, quite ad hoc, terminology:

Definition 2.3. Let C be a closed subset of $[0,1]^m$. We say that C is *n-fat* if there is a definable map F from C to $[0,1]^n$ such that $F(C)$ contains a nonempty open subset of $[0,1]^n$. We say that C is *n-slim* if C is not n-fat.

Lemma 2.4. *Let a, b be two different real numbers. Them, there is a function $g \in M_1$ such that $g(a) = 0$ and $g(b) = 1$.*

Proof. Suppose $a < b$ (the case $b < a$ is analogous). There is a rational i/n such that $a < i/n < (i+1)/n < b$. Consider the unique function g such that $g(x) = 0$ for $0 \le x \le i/n$, $g(x) = 1$ for $(i+1)/n \le x \le 1$, and g is affine in the interval $[i/n, (i+1)/n]$. This function has the required properties. □

Lemma 2.5. *A closed subset C of $[0,1]^m$ is n-fat if and only if there is a surjective definable map from C to $[0,1]^n$.*

Proof. If the definable map from C onto $[0,1]^n$ exists, then clearly, C is n-fat. Conversely, suppose $F(C)$ has nonempty interior in $[0,1]^n$. Then $F(C)$ contains a product of n rational intervals $[a_1, b_1] \times \ldots \times [a_n, b_n]$. By Lemma 2.4, let $g_i \in M_1$ be a McNaughton function such that $g_i(a_i) = 0$ and $g_i(b_i) = 1$. Let $g'_i(x_1, \dots, x_n) = g_i(x_i)$ and $G = (g'_1, \dots, g'_n)$. Then $(G \circ F)|_C$ is a surjective definable map from C to $[0,1]^n$. □

By the previous lemma, fatness is not really a new concept; however, the fatness versus slimness terminology turns out to be useful. In fact we observe that:

Lemma 2.6.

(1) *The union of two n-slim closed subsets of $[0,1]^m$ is n-slim.*

(2) *The image of an n-slim closed subset of $[0,1]^m$ under a definable map is n-slim.*

(3) *If $m < n$, then $[0,1]^m$ is n-slim.*

Proof. For the first point, let C, D be two n-slim closed subsets. Suppose for an absurdity $C \cup D$ is n-fat. Then there is a definable map F such that $F(C \cup D)$ contains an open subset O of $[0,1]^n$. Note $F(C \cup D) = F(C) \cup F(D)$. Hence we have $O \subseteq F(C) \cup F(D)$. Since $F(C)$ is closed, $O \setminus F(C)$ is an open subset of $[0,1]^n$, and it is nonempty, otherwise O would be included in $F(D)$ and D would be n-fat. So C is n-fat, contrary to the n-slimness of C. Thus $C \cup D$ is n-slim.

For the second point, let C be closed in $[0,1]^m$ and be n-slim. Let F be a definable map and $D = F(C)$. Suppose for an absurdity that D is n-fat. Then there is a definable map F' such that $F'(D)$ contains an open set in $[0,1]^n$. So, the image of C under the definable map $F' \circ F$ contains an open set in $[0,1]^n$, contrary to the slimness of C. Thus, D is also n-slim.

For the third point, suppose for an absurdity that $[0,1]^m$ is n-fat. Then there is a definable map $F = (f_1, \ldots, f_n)$ such that $F([0,1]^m)$ has nonempty interior in $[0,1]^n$ and is a rational polyhedron. Now the set T of tuples $t = (g_1, \ldots, g_n)$, such that g_i is an affine constituent of f_i, is finite. For some $t \in T$, $t([0,1]^m)$ must have nonempty interior, because the union of a finite set of polyhedra with empty interior has empty interior. So, we have a tuple of affine functions $t \in T$ such that $t([0,1]^m)$ has nonempty interior in $[0,1]^n$. Since $m < n$, this is impossible by elementary linear algebra considerations. $\square$

As a main result of the paper, we give a characterization of the n-tuples of C_n which generate a copy of M_n. We will use the following lemma.

Lemma 2.7. *Let $H = (h_1, \ldots, h_m)$ be a m-tuple of functions from $[0,1]^n$ to $[0,1]$. The subalgebra of $[0,1]^{[0,1]^n}$ generated by $\sim H$ is isomorphic to $M_m|_{Range(H)}$.*

Proof. The subalgebra $\langle\sim H\rangle$ generated by $\sim H$ is the set $\{g(h_1,\ldots,h_m)|g \in M_m\}$. Consider the map ϕ sending $g \in M_m$ to $g(h_1,\ldots,h_m)$. Indeed, ϕ is clearly surjective, and $\phi(g) = \phi(g')$ occurs if and only if g and g' coincide on the range of H. So, ϕ gives an isomorphism from $M_m|_{Range(H)}$ to $\langle\sim H\rangle$. □

Theorem 2.8. *Given an n-tuple H of elements of C_n, $\sim H$ generates a copy of M_n if and only if H, considered as a function from $[0,1]^n$ to itself, is surjective.*

Proof. Suppose H is surjective. Then $Range(H) = [0,1]^n$ and $M_n|_{Range(H)} = M_n$. By Lemma 2.7, the MV-algebra generated by $\sim H$ is isomorphic to M_n.

(We acknowledge [10] for the implication from left to right). We suppose that H is not surjective. Since H is continuous, $Range(H)$ is a proper closed subset of $[0,1]^n$. For some natural number p, the number of rationals in $Range(H)$ with denominator p is less than the number of rationals in $[0,1]^n$ with denominator p. So, the number of maximal ideals in $M_n|_{Range(H)}$ with rank p is less than the number of maximal ideals of M_n with rank p. Thus, the two MV-algebras cannot be isomorphic (recall that a maximal ideal M in an MV-algebra A has rank p if A/M has $p+1$ elements). □

Proposition 2.9. *If $m < n$, then no m-tuple of functions of C_n can generate an MV-algebra containing a copy of M_n.*

Proof. Let A be an MV-algebra generated by m functions $f_1,\ldots,f_m$. Then the range of $(f_1,\ldots,f_m)$ is n-slim, and also the range of any tuple of elements of A is n-slim, by Lemma 2.6. Suppose there is an isomorphism ϕ from M_n to a subalgebra of A. Let $l_i = \phi(\pi_i)$. Then the range of $(l_1,\ldots,l_n)$ is n-slim whereas the range of $(\pi_1,\ldots,\pi_n)$ is n-fat. So the range of $(\pi_1,\ldots,\pi_n)$ is not contained in the range of $(l_1,\ldots,l_n)$. By Lemma 3.4, there is a function $f \in M_n$ such that $f \circ (l_1,\ldots,l_n)$ is identically zero but $f \circ (\pi_1,\ldots,\pi_n)$ is not identically zero. So ϕ cannot exist. □

Corollary 2.10. *For every $m < n$, M_m does not contain any isomorphic copy of M_n.*

Proof. This is because for $m < n$, every n-tuple in M_m has an n-slim image. □

Note that M_n, instead, may contain proper copies of itself. For example, consider the subalgebra of M_1 generated by $x \oplus x$. Moreover,

Proposition 2.11. *C_n contains a copy of M_m for every m, n.*

Proof. We can suppose $m > n$. Peano in [13] constructed a continuous surjective function from $[0, 1]$ to $[0, 1]^2$. Adding dummy variables, one obtains a continuous surjective function from $[0, 1]^n$ to $[0, 1]^{n+1}$ for every n. By composition, we obtain a continuous surjective function F from $[0, 1]^n$ to $[0, 1]^m$ for every $m > n$. Write $F = (f_1, \ldots, f_m)$. The range of F is $[0, 1]^m$. Let A be the subalgebra of C_n generated by F, and $\phi : M_m \to A$ be the function such that $\phi(f) = f \circ F$. Then ϕ is an injective homomorphism. Hence, C_n contains a copy of M_m. □

Note that, the construction above provides a *canonical* copy of M_m in C_n for *every* m, n. For instance, consider $m = 2$ and $n = 1$. Let S be the continuous surjective function from $[0, 1]$ to $[0, 1]^2$ given in [13]. Write $S = (S_1, S_2)$. Then S_1 and S_2 generate a copy of M_2 in C_1.

3 Hopfian MV-algebras

In this section we give another proof of the implication from left to right of Theorem 2.8. The proof, albeit lengthy, has the advantage of introducing a notion of universal algebra which is not yet sufficiently exploited in the MV-algebra literature, but it seems promising. This notion is Hopfianity.

Definition 3.1. An algebraic structure A is called Hopfian if every surjective endomorphism of A is an automorphism.

Hopfianity was born in group theory, in relation with the fundamental groups of surfaces studied by Hopf, see [6], but it makes perfect sense in universal algebra.

Now we continue with the following lemma of universal algebra, for which we acknowledge [16]:

Lemma 3.2. *Let V be a variety with finitary operations generated by finite algebras. Let F be a free finitely generated object of V. Then F is Hopfian. Moreover, let X be a minimal cardinality generating set of F. Then X is a free basis of F.*

Proof. Since V is generated by finite algebras, the relatively free algebras in V are residually finite (the homomorphisms into the finite algebras generating V separate points). Any finitely generated, residually finite universal algebra (with finitary operations) is Hopfian by a theorem of Malcev (see [5]). So F is Hopfian.

Now suppose X is a minimal cardinality finite generating set for F. Let Y be a free basis. It must have at least as many elements as X, so we can choose an onto map from Y to X. This must extend to a surjective endomorphism from F to F, which must be an automorphism since F is Hopfian. But then our onto map from Y to X is one-one, so X is a free basis. □

Note that the variety of MV-algebras is generated by finite algebras, so the proof of the previous lemma implies that M_n is Hopfian for every n. This was also proven in [12], where more generally we find that

Theorem 3.3. (see [12]) *Every finitely presented MV-algebra is Hopfian.*

We continue with a lemma:

Lemma 3.4. *Let C, D be two closed subsets of $[0,1]^n$ such that C is not included in D. Then there is a function $f \in M_n$ which is identically zero on D but not identically zero on C.*

Proposition 3.5. *Every n-tuple of elements of M_n which generates M_n is surjective.*

Proof. First, we show that *no* $n-1$-tuple of functions $h_1, \ldots, h_{n-1}$ (possibly with repetitions) generates M_n. In fact, otherwise every n-tuple of elements of M_n would have the form $g_1(h_1, \ldots, h_{n-1}), \ldots, g_n(h_1, \ldots, h_{n-1})$, where $g_i \in M_n$. By Lemma 2.6, the image of any such tuple is n-slim, whereas the image of $\pi_1, \ldots, \pi_n$ is n-fat (it is the whole $[0,1]^n$).

Hence, every n-tuple $H = (h_1, \ldots, h_n)$ of elements which generate M_n is a minimal cardinality generating set. The variety of MV-algebras is generated by finite MV-algebras. Then by the previous lemma, $\sim H$ is a free basis of M_n. So there is an automorphism α of M_n sending h_i to π_i. If H were not surjective, then by Lemma 3.4 there would be a function $f \in M_n$ such that $f(h_1, \ldots, h_n)$ is the identically zero function, whereas $f(\pi_1, \ldots, \pi_n)$ is not the identically zero function. So α could not exist. Hence, H is surjective. □

From Proposition 3.5 follows the implication from left to right of the Theorem 2.8, that is a non-surjective n-tuple H of functions in C_n cannot generate a copy of M_n. In fact, suppose for an absurdity that $H = (h_1, \ldots, h_n)$ is non-surjective. If the subalgebra of H is isomorphic to M_n via an isomorphism ϕ, then ϕ sends H, which is not surjective, to a generating n-tuple G of M_n, which is surjective by the previous proposition. By Lemma 3.4, there is a function $f \in M_n$ such that $f(h_1, \ldots, h_n)$ is identically zero but $f(g_1, \ldots, g_n)$ is not identically zero. So, ϕ cannot exist.

Lemma 3.6. *Let C be a closed subset of $[0,1]^n$. Suppose $M_n|_C$ is isomorphic to M_n. Then $C = [0,1]^n$.*

Proof. Suppose there is an isomorphism $\iota : M_n|_C \to M_n$. Let $r : M_n \to M_n|_C$ be the restriction map. The kernel of r is the ideal $I(C) = \{f \in M_n | f(x) = 0 \ for\ every\ x \in C\}$. Then $\iota \circ r$ is a surjective endomorphism of M_n. Since M_n is Hopfian, $\iota \circ r$ is an isomorphism, so r is an isomorphism, and its kernel is $I(C) = 0$, and this implies $C = [0,1]^n$. □

[10] poses also the problem whether Theorem 2.8 still holds if H is any n-tuple of functions from $[0,1]^n$ to $[0,1]$, not necessarily continuous. The answer is given in the following corollary.

Corollary 3.7. *Let H be an n-tuple of functions from $[0,1]^n$ to $[0,1]$, not necessarily continuous. Then $\sim H$ generates a copy of M_n (in the MV-algebra $[0,1]^{[0,1]^n}$) if and only if $Range(H)$ is dense in $[0,1]^n$.*

Proof. If $Range(H)$ is dense then $M_n|_{Range(H)}$ is isomorphic to $M_n|_{[0,1]^n}$ which is M_n.

Conversely, if $Range(H)$ is not dense, then it has a closure C, and $M_n|_{Range(H)}$ is isomorphic to $M_n|_C$. So, the MV-algebra generated by $\sim H$ is isomorphic to $M_n|_C$. If this last MV-algebra is isomorphic to M_n, then by the previous lemma, this implies $C = [0,1]^n$, so $Range(H)$ is dense, contrary to the hypothesis. □

We have several examples of Hopfian MV-algebras. First, every simple MV-algebra is Hopfian. More interestingly, by [12], every finitely presented MV-algebra is Hopfian, and there is a proper class of Hopfian Boolean algebras by [7].

We add to the collection a seemingly new family of examples in:

Theorem 3.8. *Let us denote with S_m the finite MV-chain with $m+1$ elements. Every MV-algebra A which is an infinite product of finite different MV-chains of the form S_{2^i} is Hopfian.*

Proof. It is enough to prove that the only surjective homomorphism h from a product A of finite different MV-chains S_{2^i} (where $i \in I$ and $I \subseteq \omega$) to a finite MV-chain S_k is the $i-th$ projection, assuming that $k = 2^i$ for some $i \in I$.

In order to prove the claim above, note that A contains at least one sequence

$$s = (0, 0, 0, \dots, 0, 1/2^k, 1/2^k \dots),$$

where the first m components are zero and all the others are $1/2^k$.

If $h(s) = 0$, then

$$h(2^k s) = h(0, 0, 0, \dots, 1, 1, 1, \dots) = 0,$$

hence h depends only on the first m components. That is, $h(x_1, x_2, \dots) = k(x_1, x_2, \dots, x_m)$, where k is a surjective function from a finite MV-algebra of the form $S_{n_1} \times \dots \times S_{n_m}$ to S_k. But every surjective function from a finite MV-algebra to a finite MV-chain is a projection.

If instead $h(s) \neq 0$ then $h(s) = a/b$, where $b \leq k \leq 2^k - 1$, and hence $h((2^k - 1)s) = 1$. By complementation we have

$$h(\neg(2^k - 1)s) = h(1, 1, 1, \dots, 1/2^k, 1/2^k \dots) = 0,$$

and, since $s \leq \neg((2^k - 1)s)$, we have $h(s) = 0$. But this is absurd, since $h(s) \neq 0$. $\square$

We note also that:

Theorem 3.9. *The class of all Hopfian MV-algebras cannot be axiomatized in first order logic.*

Proof. Suppose Hopfianity is axiomatized by a first order theory T. Since T is valid on finite structures, it should be valid also on pseudofinite MV-algebras. Now $\{0, 1\}^\omega$ is pseudofinite by [1], but it is not Hopfian, because of the Bernoulli shift sending $x_0, x_1, x_2, x_3 \dots$ to $x_1, x_2, x_3, \dots$ (in [1], pseudofinite MV-algebras are erroneously called hyperfinite). $\square$

By using the Bernoulli shift we also can prove that:

Theorem 3.10. *For every infinite MV-algebra A there is a non-Hopfian MV-algebra B such that A is a subalgebra of B and B has the same cardinality as A.*

Proof. Let B be the set of all countable sequences of elements of A which are constant except for a finite number of components. $\square$

4 A categorial theorem

In this section we prove a category theoretic theorem which generalizes Theorem 2.8.

Lemma 4.1. (see [11]) *The image of a rational polyhedron P under a definable map F is a rational polyhedron.*

Lemma 4.2. *Let $C \subseteq [0,1]^m, D \subseteq [0,1]^n$ be two closed sets. Then $M_m|_C$ embeds in $M_n|_D$ if and only if there is a surjective definable map from D to C.*

Proof. Let F be a definable map from D onto C. Then the function from f to $f \circ F$ is an injective homomorphism from $M_m|_C$ to $M_n|_D$.

Conversely, suppose that $M_m|_C$ embeds in $M_n|_D$. Call j the embedding.

Let us consider the definable map g from D to C given simply by the counterimage map j^{-1} between the maximal spaces of the two MV-algebras. This map is surjective. In fact, let I be a maximal ideal of $M_m|_C$. Since j is injective, $j(I)$ is a proper ideal of $M_n|_D$. By Zorn's Lemma there is a maximal ideal M in $M_n|_D$ such that $j(I) \subseteq M$. Then $I \subseteq j^{-1}(M)$ and, since I is maximal, $I = j^{-1}(M)$. So, g is a surjective definable map from D to C. $\square$

Lemma 4.3. *Let A, B be two finitely generated, semisimple MV-algebras. Then there is a surjection from A to B if and only if there is a definable homeomorphism from $Max(B)$ to a subset of $Max(A)$.*

Proof. Suppose that A and B are semisimple, A is generated by n elements, and B is generated by m elements. Then A is isomorphic to $M_n|_{Max(A)}$ and B is isomorphic to $M_m|_{Max(B)}$.

Suppose there is a surjection from A to B. Then by [11, Lemma 3.12] there is a definable homeomorphism from $Max(B)$ to a subset of $Max(A)$.

Conversely, suppose that j is a definable homeomorphism from $Max(B)$ to a subset of $Max(A)$. Then B is isomorphic to $M_n|_{j(Max(B))}$. Consider the map s sending $f \in M_n|_{Max(A)}$ to $f|_{j(Max(B))} \in M_n|_{j(Max(B))}$. Every function $g \in M_n|_{j(Max(B))}$ is a definable map, so it can be extended to a definable map on $Max(A)$. This means that the map s is surjective. So there is a surjection from A to B. $\square$

Lemma 4.4. (see [9]) *Let C be a closed subset of $[0,1]^m$ and D be a closed subset of $[0,1]^{m'}$. Then $M_m|_C$ is isomorphic to $M_{m'}|_D$ if and only if C and D are definably homeomorphic.*

Lemma 4.5. *Let H be an m-tuple in C_n and let K be an m'-tuple in $C_{n'}$. The subalgebras generated by $\sim H$ and $\sim K$ are isomorphic if and only if their ranges are definably homeomorphic.*

Proof. Let C be a closed subset of $[0,1]^m$ and let K be a closed subset of $[0,1]^{m'}$. By Lemma 4.4, $M_m|_C$ is isomorphic to $M_{m'}|_D$ if and only if C and D are definably homeomorphic.

Then $M_m|_{Range(H)}$ is isomorphic to $M_{m'}|_{Range(K)}$ if and only if the two ranges are definably homeomorphic. So, by Lemma 2.7, the algebras generated by $\sim H$ and $\sim K$ are isomorphic if and only if the ranges are definably homeomorphic. $\square$

In particular, if H, K are two m-tuples in C_n with the same range, then the subalgebras generated by $\sim H$ and $\sim K$ are isomorphic, so these subalgebras share every property invariant under MV-algebra isomorphism. Note however that H and K could have very different geometric properties, despite having the same range. For instance, H could be differentiable and K could not.

Theorem 4.6. *Consider the map ρ sending the MV-algebra generated by the set $\sim H$ associated to an m-tuple H of functions in C_n to the range of H. Then ρ is well defined up to definable homeomorphism. Moreover, ρ can be extended to a functorial duality between the following subcategories of finitely generated MV-subalgebras of C_n (with MV-algebra homomorphisms as morphisms) and closed subsets of $[0,1]^n$ up to definable homeomorphism (with definable maps as morphisms), respectively:*

(1) *The copies of* M_k *and the sets definably homeomorphic to* $[0,1]^k$.

(2) *The MV-algebras containing a copy of* M_k *and the* k*-fat sets.*

(3) *the MV-algebras embeddable in* M_k *and the sets* S *such that there is a surjective definable map from* $[0,1]^k$ *to* S.

(4) *The homomorphic images of* M_k *and the sets* S *such that there is an injective definable map from* S *to* $[0,1]^k$.

(5) *The finitely presented MV-algebras and the rational polyhedra.*

(6) *The projective MV-algebras and the Z-retracts of* $[0,1]^h$ *for some* h (*for the definition of Z-retract see* [11]).

Proof. Since the maximal space of $M_m|_{Range(H)}$ is $Range(H)$ and the maximal space of M_k is $[0,1]^k$, the first point follows from Lemma 4.5.

By Lemma 4.2, M_k embeds in $M_m|_{Range(H)}$ if and only if there is a surjective definable map from $Range(H)$ to $[0,1]^k$, that is, $Range(H)$ is k-fat. This proves the second point.

The third point again follows from Lemma 4.2, and similarly, the fourth point follows from Lemma 4.3.

For the fifth point, if $\sim H$ generates a finitely presented subalgebra A of C_n then, by [11], A is isomorphic to the restriction of M_m to a rational polyhedron P. But A is also isomorphic to the restriction of M_m to the range of H. By Lemma 2.7, the range of H is definably homeomorphic to P, and by Lemma 4.1, the range of H is itself a rational polyhedron. The converse is analogous.

For the last point, if $\sim H$ generates a projective subalgebra A of C_n, by [3], A is isomorphic to the restriction of M_m to a Z-retract P of $[0,1]^k$ for some k. But A is also isomorphic to the restriction of M_m to the range of H. By Lemma 2.7, the range of H is definably homeomorphic to P, so the range of H is itself a Z-retract of $[0,1]^k$. The converse is analogous. □

5 Conclusions

We have seen that C_n contains many copies of M_n, besides the standard McNaughton model. It remains to understand, more generally, the structure of the isomorphic copies of finitely generated, or at least finitely presented, MV-algebras. In fact, the theorem of [11] mentioned above says that every finitely presented MV-algebra is isomorphic to the MV-algebra of the restrictions of the McNaughton functions to a rational polyhedron. This gives linear, standard models for finitely presented MV-algebras, but surely there are many other nonlinear models of these MV-algebras, and they deserve to be studied. Likewise for projective MV-algebras, etc.

As shown in Section 3, the notion of Hopfianity could be an interesting approach and a useful tool to study MV-algebras. We believe that Hopfianity is a property which deserves to be studied, inside and outside MV-algebras.

To conclude, the idea of finding nonlinear models of algebras can be used in other contexts than MV-algebras, for instance Riesz MV-algebras, ℓ-groups, etc., stressing the strong suitability with many applications, for instance, Artificial Neural Networks.

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