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Equivalences in Bicategories

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Abstract. In this paper, we establish some connections between the concept of an equivalence of categories and that of an equivalence in a bicategory. Its main result builds upon the observation that two closely related concepts, which could both play the role of an equivalence in a bicategory, turn out not to coincide. Two counterexamples are provided for that goal, and detailed proofs are given. In particular, all calculations done in a bicategory are fully explicit, in order to overcome the difficulties which arise when working with bicategories instead of 2-categories.

1 Introduction and Preliminaries

The work deals with equivalences as 1-cells in a bicategory. Therefore, we refer the reader to [4], [5], or [8] for an exact and detailed definition of bicategories. For other basic categorical definitions, the reader could consult, for example, [5] or [9]. Below, we will give the definition of an equivalence in a bicategory similar to that given in [10], followed by further definitions essentially given in [2] (see also [6] or [7]).

Two sections are provided for this purpose. The first one introduces the main definition of an equivalence in a bicategory and some auxiliary defi-

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nitions. The second section provides new characterizations of such equivalences, in order to get this, at about the end of this paper, a result which essentially makes the link between the definition provided for an equivalence in a bicategory and that of an equivalence as a classical functor between two categories (Proposition 3.6). We then immediately provide two counterexamples to show that the two close definitions used in the literature, of an equivalence in a bicategory, do not coincide (Theorem 3.7).

Some new definitions, like a conjugate of an equivalence (as a 1-cell in a bicategory), co-fullness, and co-faithfulness have been introduced to facilitate this presentation.

At the end of the paper, we mention an idea to how to internalize the concept of equivalences in bicategories.

2 Equivalences in bicategories

In this section we provide a definition of an equivalence in a bicategory and we give different implications that can have this definition in connection with other classic concepts.

Definition 2.1. Let $f\colon X \to Y$ be a 1-cell in a bicategory $\mathcal{C}$. We say that f is

(i) *full* if, for every object $C \in \mathcal{C}$, the functor

$$\begin{array}{rrcl} \mathcal{C}(C,f)\colon & \mathcal{C}(C,X) & \to & \mathcal{C}(C,Y) \\ & h & \mapsto & f \circ h = fh \\ & (\alpha\colon k \Rightarrow h) & \mapsto & f \circ \alpha = 1_f * \alpha \end{array}$$

is full. The α is a 2-cell in the bicatory $\mathcal{C}$ and $*$ is the horizontal composition of 2-cells. We will use these notations throughout the text;

(ii) *faithful* if, for every object $C \in \mathcal{C}$, the functor

$$\mathcal{C}(C,f)\colon \mathcal{C}(C,X) \to \mathcal{C}(C,Y)$$

is faithful;

(iii) an *equivalence* if, there exist a 1-cell $f^*\colon Y \to X$ and two invertible 2-cells $\epsilon_f\colon f \circ f^* \Rightarrow 1_Y$ and $\eta_f\colon 1_X \Rightarrow f^* \circ f$. The f^* will be called a *conjugate* of f.

Remark 2.2. It follows immediately from the definition of an equivalence f that:

(i) Any conjugate of f is also an equivalence.

(ii) f is a conjugate of f^*.

(iii) Any 1-cell isomorphic to a conjugate of f is also a conjugate of f, and any 1-cell isomorphic to f is also an equivalence.

(iv) Any two conjugates of f are isomorphic.

Example 2.3. (i) For every $X \in \mathcal{C}$, 1_X is an equivalence. More generally, if f is invertible, then it is an equivalence, since f^{-1} is an evident conjugate of f. In fact, the inverse of f, if it exists, is the strong version of the conjugate of f, which is sometimes called, in the literature, "quasi inverse" of f (see [10, Definition 16]).

(ii) An equivalence between two categories is an equivalence as a 1-cell in the 2-category **Cat** (2-cells are the natural transformations between functors).

Let us see some consequences emerging from the above definitions.

Proposition 2.4. *If $f\colon X \to Y$ is an equivalence, then it is full and faithful.*

Proof. We denote by

$$\begin{aligned} a_{hgf}\colon \quad (hg)f &\Rightarrow h(gf) \\ r_f\colon \quad f \circ 1_X &\Rightarrow f \\ l_f\colon \quad 1_Y \circ f &\Rightarrow f, \end{aligned}$$

respectively, the natural isomorphisms of, associativity of composition, right and left units in a bicategory.
f is faithful: Let us assume that there exist two 2-cells $\alpha, \gamma\colon g \Rightarrow h$ such that $f \circ \alpha = \beta = f \circ \gamma$ and let us show $\alpha = \gamma$. We have

$$f^* \circ (f \circ \alpha) = f^* \circ (f \circ \gamma). \tag{1}$$

By the naturality of the isomorphisms a_{fgh}, introduced above, we get

$$[(f^* \circ f) \circ \alpha] \circ a_{gff^*} = a_{hff^*} \circ [f^* \circ (f \circ \alpha)]. \tag{2}$$

Using (1) and (2), we get

$$a^{-1}_{hff^*} \circ [(f^* \circ f) \circ \alpha] \circ a_{gff^*} = a^{-1}_{hff^*} \circ [(f^* \circ f) \circ \gamma] \circ a_{gff^*}$$

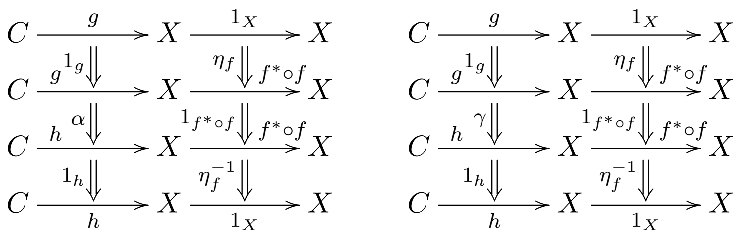
or

$$(f^* \circ f) \circ \alpha = (f^* \circ f) \circ \gamma,$$

which implies

$$(\eta_f \star 1_h) \circ (1_{f^* \circ f} \star \alpha) \circ (\eta_f^{-1} \star 1_g) = (\eta_f \star 1_h) \circ (1_{f^* \circ f} \star \gamma) \circ (\eta_f^{-1} \star 1_g). \quad (3)$$

The equality (3) is exactly the equality between the vertical compositions of the two diagrams



By applying the Interchange law (coming from the fact that the horizontal composition is a functor between hom-categories) to both members of the equality (3) (or directly to the two diagrams), we get

$$1_{1_X} \star \alpha = 1_{1_X} \star \gamma,$$

and so

$$\alpha = \gamma.$$

f is full: If $g, h \colon C \to X$ are two 1-cells and $\beta \colon f \circ g \Rightarrow f \circ h$ is a 2-cell, we shall find $\alpha \colon g \Rightarrow h$ such that $f \circ \alpha = \beta$. The α will be the composition of the 2-cells

$$\begin{array}{ccccc}
C & & \xrightarrow{\qquad g \qquad} & & X\\
 & & l_g^{-1}\Downarrow & & \\
C & \xrightarrow{g} & X & \xrightarrow{1_X} & X\\
 & & \eta_f\circ g \Downarrow & & \\
C & \xrightarrow{g} & X & \xrightarrow{f^*\circ f} & X\\
 & & a_{f^*fg}\Downarrow & & \\
C & \xrightarrow{f\circ g} & Y & \xrightarrow{f^*} & X\\
 & & f^*\circ\beta\Downarrow & & \\
C & \xrightarrow{f\circ h} & Y & \xrightarrow{f^*} & X\\
 & & a^{-1}_{f^*fh}\Downarrow & & \\
C & \xrightarrow{h} & X & \xrightarrow{f^*\circ f} & X\\
 & & \eta_f^{-1}\circ h\Downarrow & & \\
C & \xrightarrow{h} & X & \xrightarrow{1_X} & X\\
 & & l_h \Downarrow & & \\
C & & \xrightarrow[\qquad h \qquad]{} & & X.
\end{array}$$

We need to check that $f\circ\alpha = \beta$. Let us consider the equivalence between the following equalities:

$$f\circ\alpha = \beta \tag{4}$$

$$f^*\circ(f\circ\alpha) = f^*\circ\beta \tag{5}$$

$$(f^*\circ f)\circ\alpha = a^{-1}_{f^*fh}\circ(f^*\circ\beta)\circ a_{f^*fg} \tag{6}$$

$$(\eta_f^{-1}\circ h)\circ((f^*\circ f)\circ\alpha) = (\eta_f^{-1}\circ h)\circ a^{-1}_{f^*fh}\circ(f^*\circ\beta)\circ a_{f^*fg} \tag{7}$$

$$l_h\circ(1_X\circ\alpha) = l_h\circ(\eta_f^{-1}\circ h)\circ a^{-1}_{f^*fh}\circ(f^*\circ\beta)\circ a_{f^*fg}\circ(\eta_f\circ g) \tag{8}$$

$$\alpha\circ l_g = l_h\circ(\eta_f^{-1}\circ h)\circ a^{-1}_{f^*fh}\circ(f^*\circ\beta)\circ a_{f^*fg}\circ(\eta_f\circ g) \tag{9}$$

$$\alpha = l_h\circ(\eta_f^{-1}\circ h)\circ a^{-1}_{f^*fh}\circ(f^*\circ\beta)\circ a_{f^*fg}\circ(\eta_f\circ g)\circ l_g^{-1}, \tag{10}$$

which is exactly the expression of α emanating from the above diagram.
(5) ⇔ (6), by the naturality of the isomorphisms a_{fgh},
(7) ⇔ (8), by the Interchange law,
(8) ⇔ (9), by the naturality of the isomorphisms l_f. □

The converse of the last proposition is not always true (see Theorem 3.7), but we immediately have the following consequence.

Corollary 2.5. *For $f : X \to Y$, if there exists a 1-cell $f^* : Y \to X$ and an isomorphism $\eta_f\colon 1_X \Rightarrow f^*\circ f$, then f is full and faithful.*

Proof. It suffices to check in the proof of Proposition 2.4 that we have not used the isomorphism $\epsilon_f\colon f\circ f^* \Rightarrow 1_Y$. □

Many interesting properties can be deduced from the previous results.

Proposition 2.6. *The following two statements are equivalent:*

(1) *f is full, faithful, there exists a 1-cell f^* and an isomorphism $\epsilon_f \colon f \circ f^* \Rightarrow 1_Y$.*

(2) *For every $C \in \mathcal{C}$, $\mathcal{C}(C, f)$ is an equivalence.*

Proof. (1) ⇒(2): To show this implication it suffices to prove that for every C and for every $g : C \to Y$, there exists $h : C \to X$ such that $f \circ h \cong g$. But this results immediately from the sequence of the evident isomorphisms

$$g \cong 1_Y \circ g \cong (f \circ f^*) \circ g \cong f \circ (f^* \circ g)$$

and by taking $h = f^* \circ g$.

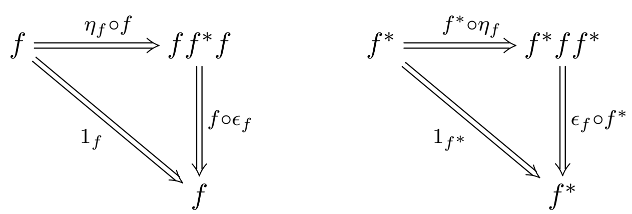
(2)⇒(1): Clearly, to show this implication, it is enough to show the existence of $f^* : Y \to X$ and an isomorphism $\epsilon_f \colon f \circ f^* \Rightarrow 1_Y$. We may take $C = Y$. Then $\mathcal{C}(Y, f)$ is an equivalence, and so for $1_Y : Y \to Y$, there exists $f^* : Y \to X$ such that $1_Y \cong f \circ f^*$. □

By Proposition 2.4, we get the following corollary.

Corollary 2.7. *If f is an equivalence, then for every $C \in \mathcal{C}$, $\mathcal{C}(C, f)$ is an equivalence.*

Remark 2.8. (i) The converse of Corollary 2.7 is not always true. In fact, we will prove later that we need more conditions on f for it to be an equivalence.

(ii) In the definition of an equivalence, if in addition the two isomorphisms η_f and ϵ_f are natural, then one can choose those isomorphisms so that the usual triangular identities



are verified. To prove this, the reader could consider, for example, the idea used in the proof of Proposition 27 in [3].

3 New characterizations

Via the new concepts of co-full and co-faithful 1-cells, we will establish the main result of this work, Theorem 3.7, which provides two counterexamples against the equivalence of two apparent close concepts of an equivalence in a bicategory.

Definition 3.1. Let $f\colon X \to Y$ be a 1-cell in a bicategory $\mathcal{C}$. We say that f is

(i) *co-full* if, for every object $C \in \mathcal{C}$, the functor

$$\begin{array}{rrcl} \mathcal{C}(f,C)\colon & \mathcal{C}(Y,C) & \to & \mathcal{C}(X,C) \\ & h & \mapsto & h \circ f = hf \\ & (\alpha\colon k \Rightarrow h) & \mapsto & \alpha \circ f = \alpha * 1_f \end{array}$$

is full,

(ii) *co-faithful* if, for every object $C \in \mathcal{C}$, the functor

$$\mathcal{C}(f,C)\colon \mathcal{C}(Y,C) \to \mathcal{C}(X,C)$$

is faithful.

Proposition 3.2. *If $f\colon X \to Y$ is an equivalence, then it is co-full and co-faithful.*

Proof. Straightforward, by using the same arguments used in the proof of Proposition 2.4 and changing η_f by ϵ_f. □

We need less conditions on f to have the result given above.

Corollary 3.3. *For $f : X \to Y$, if there exists a 1-cell f^* and an isomorphism $\epsilon_f\colon f \circ f^* \Rightarrow 1_Y$, then f is co-full and co-faithful.*

Proof. Notice that we have no need of the isomorphism $\eta_f\colon 1_X \Rightarrow f^* \circ f$, in the proof of Proposition 3.2. □

Proposition 3.4. *The following two statements are equivalent:*

(1) *f is co-full, co-faithful, there exist a 1-cell $f^\star$ and an isomorphism $\eta_f\colon 1_X \Rightarrow f^* \circ f$.*

(2) *For every $C \in \mathcal{C}$, $\mathcal{C}(f,C)$ is an equivalence.*

Proof. See the proof of Proposition 2.6. □

Corollary 3.5. *If f is an equivalence, then for every $C \in \mathcal{C}$, $\mathcal{C}(f, C)$ is an equivalence.*

Proof. Use Corollary 3.3. □

Now we are able to give a new characterization of an equivalence as a 1-cell, by means of equivalences as functors.

Proposition 3.6. *The following two statements are equivalent:*
(1) *f is an equivalence.*
(2) *For every $C \in \mathcal{C}$, $\mathcal{C}(C, f)$ and $\mathcal{C}(f, C)$ are equivalences.*

Proof. (1) ⇒(2): Results immediately from Corollareies 2.7 and 3.5.
(2)⇒(1): By Propositions 2.6 and 3.4. □

At this stage we have enough ingredients to prove relatively readily the item (i) of Remark 2.8. This will be the main result of the present paper.

Theorem 3.7. *The two parts of the item* (2) *of the above proposition, no one implies the other in general.*

Proof. We shall give counterexamples to show that each of the following two equivalences are false in general:

(a) f is an equivalence if and only if for every $C \in \mathcal{C}$, $\mathcal{C}(f, C)$ is an equivalence.

(b) f is an equivalence if and only if for every $C \in \mathcal{C}$, $\mathcal{C}(C, f)$ is an equivalence.

By Proposition 3.6, this amounts to give counterexamples to show that each of the following two implications is false in general:

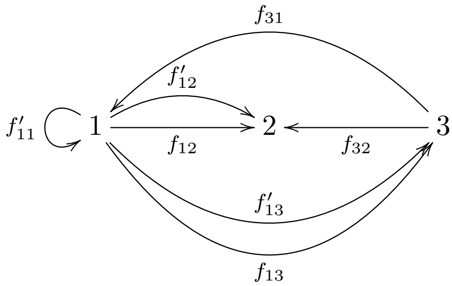
(a′) for every $C \in \mathcal{C}$, if $\mathcal{C}(C, f)$ is an equivalence then, for every $C \in \mathcal{C}$, $\mathcal{C}(f, C)$ is an equivalence.

(b′) for every $C \in \mathcal{C}$, if $\mathcal{C}(f, C)$ is an equivalence then, for every $C \in \mathcal{C}$, $\mathcal{C}(C, f)$ is an equivalence.

For (a′), consider the following bicategory $\mathcal{C}$:

- 0-cells: $\{1,2,3\}$,
- 1-cells: $f_{ii} = id_i$ for all $i \in \{1,2,3\}$ and

$$1 \overset{f'_{11}}{\to} 1;\ 1 \overset{f_{12}}{\to} 2;\ 1 \overset{f'_{12}}{\to} 2;\ 1 \overset{f_{13}}{\to} 3;\ 1 \overset{f'_{13}}{\to} 3;\ 3 \overset{f_{31}}{\to} 1;\ 3 \overset{f_{32}}{\to} 2$$



The composition for those 1-cells are defined as follows:

$$\begin{aligned}
f_{12} \circ f_{31} &= f_{32}; & f'_{12} \circ f_{31} &= f_{32};\\
f_{32} \circ f_{13} &= f'_{12}; & f_{32} \circ f'_{13} &= f'_{12};\\
f_{13} \circ f_{31} &= f_{33}; & f'_{13} \circ f_{31} &= f_{33};\\
f_{31} \circ f_{13} &= f'_{11}; & f_{31} \circ f'_{13} &= f'_{11};\\
f_{12} \circ f'_{11} &= f'_{12}; & f'_{12} \circ f'_{11} &= f'_{12};\\
f_{13} \circ f'_{11} &= f'_{13}; & f'_{13} \circ f'_{11} &= f'_{13};\\
f'_{11} \circ f_{31} &= f_{31}; & f'_{11} \circ f'_{11} &= f'_{11}.
\end{aligned}$$

We can easily check the associativity of the compositions.

- 2-cells: The 2-cells in $\mathcal{C}$ are defined as

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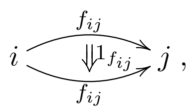

  whenever f_{ij} is defined for $i, j \in \{1,2,3\}$. Those are the invertible 2-cells.

- The non invertible 2-cells are:
  $\alpha_{11} : f_{11} \Rightarrow f'_{11}$; $\beta_{11} : f'_{11} \Rightarrow f_{11}$; $\gamma_{11} : f_{11} \Rightarrow f_{11}(\gamma_{11} \neq 1_{f_{11}})$; $\theta_{11} : f'_{11} \Rightarrow f'_{11}(\theta_{11} \neq 1_{f'_{11}})$, with the following vertical composition rules:

$$\begin{array}{rcl} \beta_{11} \circ \alpha_{11} & = & \gamma_{11} \\ \alpha_{11} \circ \beta_{11} & = & \theta_{11} \\ \theta_{11} \circ \alpha_{11} & = & \alpha_{11} \\ \alpha_{11} \circ \gamma_{11} & = & \alpha_{11} \\ \gamma_{11} \circ \gamma_{11} & = & \gamma_{11} \\ \theta_{11} \circ \theta_{11} & = & \theta_{11} \\ \gamma_{11} \circ \beta_{11} & = & \beta_{11} \\ \beta_{11} \circ \theta_{11} & = & \beta_{11} \end{array}$$

  $\alpha_{12} : f_{12} \Rightarrow f'_{12}$; $\beta_{12} : f'_{12} \Rightarrow f_{12}$; $\gamma_{12} : f_{12} \Rightarrow f_{12}(\gamma_{12} \neq 1_{f_{12}})$; $\theta_{12} : f'_{12} \Rightarrow f'_{12}(\theta_{12} \neq 1_{f'_{12}})$, with the following vertical composition rules:

$$\begin{array}{rcl} \beta_{12} \circ \alpha_{12} & = & \gamma_{12} \\ \alpha_{12} \circ \beta_{12} & = & \theta_{12} \\ \theta_{12} \circ \alpha_{12} & = & \alpha_{12} \\ \alpha_{12} \circ \gamma_{12} & = & \alpha_{12} \\ \gamma_{12} \circ \gamma_{12} & = & \gamma_{12} \\ \theta_{12} \circ \theta_{12} & = & \theta_{12} \\ \gamma_{12} \circ \beta_{12} & = & \beta_{12} \\ \beta_{12} \circ \theta_{12} & = & \beta_{12} \end{array}$$

  and $\alpha_{13} : f_{13} \Rightarrow f'_{13}$; $\beta_{13} : f'_{13} \Rightarrow f_{13}$; $\gamma_{13} : f_{13} \Rightarrow f_{13}(\gamma_{13} \neq 1_{f_{13}})$; $\theta_{13} : f'_{13} \Rightarrow f'_{13}(\theta_{13} \neq 1_{f'_{13}})$, with the following vertical composition rules:

$$\begin{aligned}
\beta_{13} \circ \alpha_{13} &= \gamma_{13} \\
\alpha_{13} \circ \beta_{13} &= \theta_{13} \\
\theta_{13} \circ \alpha_{13} &= \alpha_{13} \\
\alpha_{13} \circ \gamma_{13} &= \alpha_{13} \\
\gamma_{13} \circ \gamma_{13} &= \gamma_{13} \\
\theta_{13} \circ \theta_{13} &= \theta_{13} \\
\gamma_{13} \circ \beta_{13} &= \beta_{13} \\
\beta_{13} \circ \theta_{13} &= \beta_{13}.
\end{aligned}$$

The horizontal composition rules of those 2-cells are:

$$\begin{array}{llll}
\alpha_{12} \star \alpha_{11} = \alpha_{12}; & f_{12} \circ \gamma_{11} = \gamma_{12}; & f'_{12} \circ \gamma_{11} = \theta_{12}; & \alpha_{12} \circ f'_{11} = \theta_{12}; \\
\beta_{12} \star \beta_{11} = \beta_{12}; & f_{12} \circ \theta_{11} = \theta_{12}; & f'_{12} \circ \theta_{11} = \theta_{12}; & \beta_{12} \circ f'_{11} = \theta_{12}; \\
\alpha_{12} \star \beta_{11} = \theta_{12}; & f_{12} \circ \alpha_{11} = \alpha_{12}; & f'_{12} \circ \alpha_{11} = \theta_{12}; & \alpha_{12} \circ f'_{11} = \theta_{12}; \\
\beta_{12} \star \alpha_{11} = \theta_{12}; & f_{12} \circ \beta_{11} = \beta_{12}\ ; & f'_{12} \circ \beta_{11} = \theta_{12}; & \beta_{12} \circ f'_{11} = \theta_{12}; \\
\alpha_{13} \star \alpha_{11} = \alpha_{13}; & f_{13} \circ \gamma_{11} = \gamma_{13}; & f'_{13} \circ \gamma_{11} = \theta_{13}; & \alpha_{13} \circ f'_{11} = \theta_{13}; \\
\beta_{13} \star \beta_{11} = \beta_{13}; & f_{13} \circ \theta_{11} = \theta_{13}; & f'_{13} \circ \theta_{11} = \theta_{13}; & \beta_{13} \circ f'_{11} = \theta_{13}; \\
\alpha_{13} \star \beta_{11} = \theta_{13}; & f_{13} \circ \alpha_{11} = \alpha_{13}; & f'_{13} \circ \alpha_{11} = \theta_{13}; & \alpha_{13} \circ f'_{11} = \theta_{13}; \\
\beta_{13} \star \alpha_{11} = \theta_{13}; & f_{13} \circ \beta_{11} = \beta_{13}\ ; & f'_{13} \circ \beta_{11} = \theta_{13}; & \beta_{13} \circ f'_{11} = \theta_{13};
\end{array}$$

$\mathcal{C}$ is a 2-category: long but straightforward.

It is not hard to check that if we take $f = f_{13}$, $\mathcal{C}(C, f)$ is an equivalence for every $C \in \{1, 2, 3\}$. But, for $C = 2$ and, since the only 1-cell from 3 to 2 is f_{32}, and $f_{32} \circ f_{13} = f'_{12}$, there exists no $g : 3 \to 2$ such that $g \circ f_{13} = f_{12}$ (or more precisely $g \circ f_{13} \cong f_{12}$), then $\mathcal{C}(f_{13}, C)$ can not be essentially surjective, therefore it can not be an equivalence.

For (b′), we could take, as a counterexample, the bicategory $\mathcal{K} = \mathcal{C}^{op}$, $f = f_{31}$, and we can check easily that for every $C \in \mathcal{K}$, $\mathcal{K}(f_{31}, C)$ is an equivalence, but $\mathcal{K}(2, f_{31})$ can not be an equivalence. □

Remark 3.8. (i) As a consequence of Theorem 3.7, the authors who define the equivalence f by the property: "$\forall C \in \mathcal{C}$, $\mathcal{C}(C, f)$ is an equivalence", give a definition that can not be equivalent (in a general bicategory) to the one given in (iii) of Definition 2.1. Nevertheless, in many bicategories, the two properties p1. $\forall C \in \mathcal{C}, \mathcal{C}(C, f)$ is an equivalence,

p2. $\forall C \in \mathcal{C}$, $\mathcal{C}(C, f)$ and $\mathcal{C}(f, C)$ are equivalences,

are equivalent, and it seems to be relatively hard to find a counterexample where those two properties are not equivalent; many authors adopt the first property as a definition of equivalences.

(ii) The bicategories chosen in [2] are such that all 2-cells are invertible. Especially in that case, p1 and p2 are equivalent!

(iii) It is easy enough to see that in a 2-category $\mathcal{C}$,

$$\mathcal{C}(C, g) \circ \mathcal{C}(C, f) = \mathcal{C}(C, g \circ f) \tag{1}$$

and

$$\mathcal{C}(f, C) \circ \mathcal{C}(g, C) = \mathcal{C}(g \circ f, C) \tag{2}$$

(whenever $g \circ f$ exists), which implies directly the following properties:

P1. If f and g are faithful (co-faithful), then $g \circ f$ is faithful (co-faithful).

P2. If f and g are full (co-full), then $g \circ f$ is full (co-full).

P3. If f and g are equivalences, then $g \circ f$ is an equivalence.

Since the composition of 1-cells, in bicategories, is associative only up to an isomorphism, equalities (1) and (2) are not always true. This makes the above properties not as evident in bicategories as they are in 2-categories. Nevertheless, we still get the following results.

Proposition 3.9. *In a bicategory $\mathcal{C}$, if $f : X \to Y$ and $g : Y \to Z$ are*
P1. *faithful (co-faithful), then $g \circ f$ is faithful (co-faithful),*
P2. *full (co-full), then $g \circ f$ is full (co-full),*
P3. *equivalences, then $g \circ f$ is an equivalence.*

Proof. P1. If f and g are faithful, then $g \circ f$ is faithful: Let $\alpha_1, \alpha_2 : h \Rightarrow k$ be two 2-cells such that

$$(g \circ f) \circ \alpha_1 = (g \circ f) \circ \alpha_2. \tag{3}$$

Using the naturality of the isomorphisms a_{fgh} with α_1, we get

$$[g \circ (f \circ \alpha_1)] \circ a_{gfh} = a_{gfk} \circ [(g \circ f) \circ \alpha_1)]. \tag{4}$$

The same equation is true with α_2,

$$[g \circ (f \circ \alpha_2)] \circ a_{gfh} = a_{gfk} \circ [(g \circ f) \circ \alpha_2)]. \tag{5}$$

The equations (3), (4), and (5) imply

$$g \circ (f \circ \alpha_1) = g \circ (f \circ \alpha_2).$$

Since g is faithful,

$$f \circ \alpha_1 = f \circ \alpha_2.$$

Now, since f is also faithful, $\alpha_1 = \alpha_2$, which proves that $g \circ f$ is faithful.

We use similar arguments to show that $g \circ f$ is co-faithful if g and f are co-faithful.

P2. If f and g are full, then $g \circ f$ is full: Let us consider $\beta : (g \circ f) \circ h \Rightarrow (g \circ f) \circ k$ and find $\alpha : h \Rightarrow k$ such that $\beta = (g \circ f) \circ \alpha$. By considering β_1, which is the composite of the 2-cells

$$g \circ (f \circ h) \overset{a_{gfh}}{\Longrightarrow} (g \circ f) \circ h \overset{\beta}{\Longrightarrow} (g \circ f) \circ k \overset{a_{gfk}}{\Longrightarrow} g \circ (f \circ k), \tag{6}$$

then $\beta_1 : g \circ (f \circ h) \Rightarrow g \circ (f \circ k)$. But g is full, therefore there exists $\theta : f \circ h \Rightarrow f \circ k$ such that $g \circ \theta = \beta_1$. Since f is full, there exists $\alpha : h \Rightarrow k$ such that $\theta = f \circ \alpha$. In that way, we get

$$\beta_1 = g \circ (f \circ \alpha).$$

Using (6) we get

$$\beta = a^{-1}_{gfk} \circ \beta_1 \circ a^{-1}_{gfh}. \tag{7}$$

Using now the naturality of a_{gfk}, we get

$$[g \circ (f \circ \alpha)] \circ a^{-1}_{gfh} = a_{gfk} \circ [(g \circ f) \circ \alpha]. \tag{8}$$

(7) and (8) give

$$\beta = (g \circ f) \circ \alpha.$$

This implies $g \circ f$ is full.

Using similar arguments, we easily show that $g \circ f$ is co-full if g and f are co-full.

P3. If f and g are equivalences, then $g \circ f$ is an equivalence: To show this, it suffices to give a conjugate h of $g \circ f$. Let us consider $h = f^* \circ g^*$. We know that

$$(g \circ f) \circ h \cong g \circ (f \circ f^*) \circ g^* \cong g \circ 1_Y \circ g^* \cong g \circ g^* \cong 1_Z,$$

which form $\epsilon_{g \circ f}$.

In the other side, we have

$$h \circ (g \circ f) \cong f^* \circ (g^* \circ g) \circ f \cong f^* \circ 1_Y \circ f \cong f^* \circ f \cong 1_X,$$

which form $\eta_{g \circ f}$, and therefore h is a conjugate of $g \circ f$. □

Remark 3.10. It is well known (see, for example, [5, Proposition 8.1.4]), that if $\mathcal{C}$ is a category, then the internal categories in $\mathcal{C}$, the internal functors in $\mathcal{C}$, and the natural transformations in $\mathcal{C}$, form a 2-category. Therefore, we can speak about an equivalence for an internal functor. Evidently one may use Definition 2.1, or any other equivalent definitions given in this paper. As a perspective to this work, the reader may consider establishing the link between the definition emanating from this work applied to equivalences for internal functors and, for example, the definition given in [6]. That could be a very interesting work, which would generate many important new ideas.

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