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$\mathcal{R}L$-valued f-ring homomorphisms and lattice-valued maps

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Dedicated to Professor Bernhard Banaschewski on the occasion of his 90th Birthday

Abstract. In this paper, for each *lattice-valued map* $A \to L$ with some properties, a ring representation $A \to \mathcal{R}L$ is constructed. This representation is denoted by τ_c which is an f-ring homomorphism and a $\mathbb{Q}$-linear map, where its index c, mentions to a lattice-valued map. We use the notation $\delta^a_{pq} = (a-p)^+ \wedge (q-a)^+$, where $p, q \in \mathbb{Q}$ and $a \in A$, that is nominated as *interval projection*. To get a well-defined f-ring homomorphism τ_c, we need such concepts as *bounded*, *continuous*, and $\mathbb{Q}$-*compatible* for c, which are defined and some related results are investigated. On the contrary, we present a cozero lattice-valued map $c_\phi : A \to L$ for each f-ring homomorphism $\phi : A \to \mathcal{R}L$. It is proved that $c_{\tau_c} = c^r$ and $\tau_{c_\phi} = \phi$, which they make a kind of correspondence relation between ring representations $A \to \mathcal{R}L$ and the lattice-valued maps $A \to L$, where the mapping $c^r : A \to L$ is called a *realization* of c. It is shown that $\tau_{c^r} = \tau_c$ and $c^{rr} = c^r$.

Finally, we describe how τ_c can be a fundamental tool to extend pointfree version of Gelfand duality constructed by B. Banaschewski.

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1 Introduction

According Banaschewski's pointfree out-looking [1], the fundamental part of Gelfand duality is a pair of adjunction maps (σ, τ) between two functors $\mathfrak{C} : \mathbf{KCRFrm} \to \mathbf{SBFAnn}$ and $\mathfrak{M} : \mathbf{SBFAnn} \to \mathbf{KCRFrm}$.

In this paper, the second part of the adjunction, τ, is extended generally. It is necessary to say that for every strong f-ring A, $\tau_A : A \to \mathcal{R}\mathfrak{M}(A)$ is given by $\tau_A(a) = \hat{a}$, where $\hat{a} : \mathcal{R} \to \mathfrak{M}(A)$ is defined by $\hat{a}(p,q) = [(a-p)^+ \wedge (q-p)^+]$.

In Section 5, we define τ_c as an extension of τ_A, for a lattice-valued map $c : A \to L$ to a frame L.

Now, to analyse introductionally, we see that $p < x < q$ if and only if $\min\{(x-p)^+, (q-x)^+\}$ is nonzero, for every $p, q, x \in \mathbb{Q}$. So, for every $f \in C(X)$ and for every $p, q \in \mathbb{Q}$, $f^{-1}(p,q) = coz_X((f-p)^+ \wedge (q-f)^+)$, where the pointfree version of this equation is discussed in [2] as $\alpha(p,q) = coz_L((\alpha - p)^+ \wedge (q-\alpha)^+)$, where $\alpha \in \mathcal{R}L$ and $p, q \in \mathbb{Q}$. This is the reason for taking the notation $\delta^a_{pq} = (a-p)^+ \wedge (q-a)^+$, for every $p, q \in \mathbb{Q}$ and every $a \in A$, where A is a $\mathbb{Q}$-algebra f-ring. The above notation is fundamentally applied in construction of τ_c in this paper.

First, in the next section, we give the necessary background on frames (pointfree topology) and f-rings. Interval projections are introduced and discussed in Section 3. In Lemma 3.3, we recall the basic relations which we expect from δ^a_{pq} similar to intervals. In Section 4, we define $a_c : \mathcal{R} \to L$, given by $a_c(p,q) = c(\delta^a_{pq})$, for every $a \in A$ and $c \in F(A, L)$. The concepts of the continuous lattice-valued maps and the bounded lattice-valued maps are defined and studied, which are necessary for $a_c : \mathcal{R} \to L$ to be a frame map. Proposition 4.7 shows that there are many bounded lattice-valued map. We study continuity and bondedness of lattice-valued maps of coz_X, coz_L, ι_A and $\bar{\iota}_A$, in Propositions 4.8 and 4.10. In Section 5, the $\mathbb{Q}$-compatible lattice-valued map is defined, which is a fundamental sufficient condition for $\tau_A : A \to \mathcal{R}L$ to be an f-ring homomorphism and a $\mathbb{Q}$-linear map, which is proved in Corollary 6.9 in the last section. In Proposition 5.5 we prove that coz_X and coz_L are $\mathbb{Q}$-compatible, and in Proposition 5.7 we show that ι_A and $\bar{\iota}_A$ are $\mathbb{Q}$-compatible in the particular case of the ring of continuous functions, but the general case is left for the readers as an open problem (Remark 5.6). It is interesting to say that the set of all $\mathbb{Q}$-compatible lattice-valued maps is closed under binary meets, which is proved in Proposition 5.7.

In the last section, we introduce $\tau_c : A \to \mathcal{R}L$ given by $\tau_c(a) = a_c$. Using τ_c, we prove, in Proposition 6.10, that if c is a $\mathbb{Q}$-compatible lattice-valued map, so is c^r. We describe the relations between τ_c, c, and c^r in Proposition 6.11. Finally, we explain that τ_c is fundamental to extend pointfree version of Gelfand duality (Remark 6.12).

2 Preliminaries

Throughout the paper, all the necessary definitions and preliminary statements may be found in [1, 3, 5, 8]. Recall that a *frame* L is a complete lattice in which the infinite distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge x | x \in S\}$$

holds for all $a \in L$ and $S \subseteq L$. We denote the top element and the bottom element of L by $\top$ and $\bot$, respectively. Throughout this paper L is supposed to be a frame.

We recall from [2] that a *homomorphism* of lattice-ordered rings is, of course, a map between such rings which is, both, a ring and a lattice homomorphism. An *ℓ-ideal* in a lattice-ordered ring A is a ring ideal J of A with the added property that $|x| \leq |a|$ and $a \in J$ imply $x \in J$, for every $x, a \in A$. The set of all ℓ-ideals of A, denoted by $\mathcal{L}(A)$, is a compact frame with the partial order $\subseteq$. For $a \in A$, the ℓ-ideal generated by a is

$$[a] = \{x \in A \mid |x| \leq |a|b,\, b \geq 0 \text{ in } A\}$$

and, obviously, $[a] = [|a|]$.

Now, an f-ring is a lattice-ordered ring A which satisfies any one of the equivalent conditions

1. $(a \wedge b)c = (ac) \wedge (bc)$ for any $a, b \in A$ and $0 \leq c \in A$,
2. $|ab| = |a||b|$,
3. $[a] \cap [b] = [a \wedge b]$ for any $a, b \geq 0$ in A.

In any f-ring A, $a^2 \geq 0$ for each $a \in A$. We call an f-ring A *strong* if every $a \geq 1$ is invertible in A, and bounded if, for each $a \in A$, $|a| \leq n$ for some natural number n. Throughout this paper A is a strong f-ring.

Recall from *Frame real* that the frame $\mathcal{L}R$ of reals is obtained by taking the ordered pairs (p,q) of rational numbers as generators and imposing the relations

(R1) $(p,q) \wedge (r,s) = (p \vee r, q \wedge s)$.

(R2) $(p,q) \vee (r,s) = (p,s)$ whenever $p \leq r < q \leq s$.

(R3) $(p,q) = \bigvee\{(r,s) | p < r < s < q\}$.

(R4) $\top = \bigvee\{(p,q) |\ p,q \in \mathbb{Q}\}$.

The set $\mathcal{R}L$ of all frame homomorphisms from $\mathcal{L}R$ to L has been studied as an f-ring in [1, 2].

Corresponding to every continuous operation $\diamond : \mathbb{Q}^2 \to \mathbb{Q}$ (in particular $+, \cdot, \wedge, \vee$) we have an operation on $\mathcal{R}L$, denoted by the same symbol $\diamond$, defined by

$$\alpha \diamond \beta(p,q) = \bigvee\{\alpha(r,s) \wedge \beta(u,w) : \langle r,s\rangle \diamond \langle u,w\rangle \leq \langle p,q\rangle\},$$

where $\langle r,s\rangle \diamond \langle u,w\rangle \leq \langle p,q\rangle$ means that for each $r < x < s$ and $u < y < w$ we have $p < x \diamond y < q$.

The *cozero map* is the map $coz : \mathcal{R}L \to L$, defined by

$$coz(\alpha) = \alpha(-,0) \vee \alpha(0,-).$$

For $A \subseteq \mathcal{R}L$, let $Coz(A) = \{coz(\alpha) : \alpha \in A\}$ be the cozero part of A. The set $Coz(\mathcal{R}L)$ is denoted by $CozL$. For more details about *cozero map* and its properties which are used in this note see [2].

3 Interval projection

In this section, we introduce the notion of an *interval projection.* Interval projections are elements of a strong f-ring A, denoted by δ^a_{pq}, which are indexed by a triple (a,p,q) where $a \in A$ and $p,q \in \mathbb{Q}$. The reason for taking the notation δ^a_{pq} is that for a fixed a, interval projections δ^a_{pq} and open intervals (p,q) have the same lattice behaviour, given in Lemma 3.3. So, one can say that δ^a_{pq} are projections of open intervals (p,q) on a.

Definition 3.1. Let A be a strong f-ring. For $a \in A$ and $r, s \in \mathbb{Q}$, the element $(a-r)^+ \wedge (s-a)^+$, which we denote it by δ^a_{rs}, is called an *interval projection.*

In the following figure, in the ring of continuous functions $C([0, +\infty))$, the identity function $y = x$ and its interval projection δ^x_{pq} is drawn.

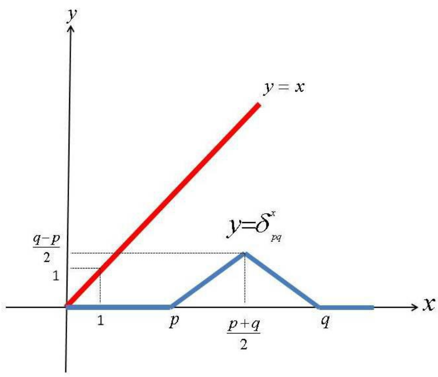

Figure 1:

Suppose that $\phi : A \to B$ is an f-ring homomorphism between two strong f-rings such that $\phi(p) = p$ for all $p \in \mathbb{Q}$. So, it is also a linear map on the field $\mathbb{Q}$, because of this, we call ϕ an f-ring homomorphism and a $\mathbb{Q}$-linear map.

Lemma 3.2. *Let $\varphi : A \to B$ be an f-ring homomorphism and a $\mathbb{Q}$-linear map. If $\varphi(1) = 1$, then*

$$\varphi(\delta^a_{pq}) = \delta^{\varphi(a)}_{pq},$$

for every $a \in A$ and $p, q \in \mathbb{Q}$.

Proof.

$$\begin{aligned}\varphi(\delta^a_{pq}) &= \varphi((a-p)^+ \wedge (q-a)^+) \\ &= (\varphi(a)-p)^+ \wedge (q-\varphi(a))^+ \\ &= \delta^{\varphi(a)}_{pq}.\end{aligned}$$

□

Lemma 3.3. *For $a \in A$ and $p, q, r, s \in \mathbb{Q}$, we have*

(1) *If $p \geq q$, then $\delta^a_{pq} = 0$,*
(2) $\delta^a_{(p\vee r)(q\wedge s)} = \delta^a_{pq} \wedge \delta^a_{rs}$,
(3) $\delta^a_{sr} \wedge \delta^a_{rq} = 0$,
(4) *If $p \leq r \leq s \leq q$, then $\delta^a_{rs} \leq \delta^a_{pq}$,*
(5) *If $p \leq r < q \leq s$, then $\delta^a_{pq} \vee \delta^a_{rs} = \delta^a_{ps}$,*
(6) *If $p \leq q$, then $\delta^a_{pq} \vee \delta^a_{qp} = \delta^a_{pq}$.*

Proof. (1) If $p \geq q$, then

$$0 \leq \delta^a_{pq} = ((a-p)^+ \wedge (q-a)^+) \leq ((a-p)^+ \wedge_c (p-a)^+) = 0.$$

Hence, $\delta^a_{pq} = 0$.

(2) Suppose $p, r, s, q \in \mathbb{Q}$. Since

$$a - (p \vee r) = (a-p) \wedge (a-r), \quad (q \wedge s) - a = (q-a) \wedge (s-a),$$

and

$$(x \wedge y)^+ = x^+ \wedge y^+,$$

we conclude that

$$\begin{aligned} \delta^a_{(p\vee r)(q\wedge s)} &= (a-(p\vee r))^+ \wedge ((q\wedge s)-a)^+ \\ &= (a-p)^+ \wedge (a-r)^+ \wedge (q-a)^+ \wedge (s-a)^+ \\ &= \delta^a_{pq} \wedge \delta^a_{rs}. \end{aligned}$$

(3) It is clear that

$$\begin{aligned} \delta^a_{sr} \wedge \delta^a_{rq} &= (a-s)^+ \wedge (a-r)^+ \wedge (r-a)^+ \wedge (q-a)^+ \\ &= (a-s)^+ \wedge \delta^a_{rr} \wedge (q-a)^+ \\ &= 0. \end{aligned}$$

(4) Let $p, r, s, q \in \mathbb{Q}$ with $p \leq r \leq s \leq q$. Then, by (2), we have

$$\delta^a_{pq} \wedge \delta^a_{rs} = \delta^a_{(p\vee r)(q\wedge s)} = \delta^a_{rs}.$$

Hence, $\delta^a_{rs} \leq \delta^a_{pq}$.

(5) Let $p, q, r, s \in \mathbb{Q}$ with $p \leq r < q \leq s$. Since $(a-p)^+ \geq (a-r)^+$ and $(s-a)^+ \geq (q-a)^+$, we conclude that

$$\begin{aligned}
\delta^a_{pq} \vee \delta^a_{rs} &= ((a-p)^+ \wedge (q-a)^+) \vee ((a-r)^+ \wedge (s-a)^+) \\
&= ((a-p)^+ \vee (a-r)^+) \wedge ((a-p)^+ \vee (s-a)^+) \\
&\quad \wedge((q-a)^+ \vee (a-r)^+) \wedge ((q-a)^+ \vee (s-a)^+) \\
&= [(a-p)^+ \wedge ((a-p)^+ \vee (s-a)^+)] \\
&\quad \wedge[((q-a)^+ \vee (a-r)^+) \wedge (s-a)^+] \\
&= (a-p)^+ \wedge (s-a)^+ \\
&= \delta^a_{ps}.
\end{aligned}$$

(6) It is clear that

$$\begin{aligned}
\delta^a_{pq} \vee \delta^a_{qp} &= [(a-p)^+ \wedge (q-a)^+] \vee [(a-q)^+ \wedge (p-a)^+] \\
&\leq [(a-p)^+ \vee (a-q)^+] \wedge [(a-p)^+ \vee (p-a)^+] \wedge \\
&\quad [(q-a)^+ \vee (a-q)^+] \wedge [(q-a)^+ \vee (p-a)^+] \\
&= (a-p)^+ \wedge [(a-p)^+ \vee (p-a)^+] \wedge \\
&\quad [(q-a)^+ \vee (a-q)^+] \wedge (q-a)^+ \\
&\leq \delta^a_{pq}.
\end{aligned}$$

Therefore, $\delta^a_{pq} \vee \delta^a_{qp} = \delta^a_{pq}$. □

4 Bounded and continuous lattice-valued maps

In this section, we define and study the concepts of the *continuous* and *bounded* lattice-valued maps. First we introduce the concept of cozero lattice-valued maps.

Definition 4.1. A mapping $c : A \to L$ is called a *lattice-valued map* on A.

We denote the set of all lattice-valued maps from a strong f-ring A into a frame L by $F(A, L)$.

Definition 4.2. A lattice-valued map $c \in F(A, L)$ is called a *cozero lattice-valued map* if it satisfies

($c1$) $c(0) = \bot$,

($c2$) if x is a unit, then $c(x) = \top$,

($c3$**)** if $x, y \geq 0$, then $c(x \vee y) = c(x) \vee c(y)$,

($c4$**)** if $x, y \geq 0$, then $c(x \wedge y) = c(x) \wedge c(y)$.

From now on, unless specified otherwise, c denotes a lattice-valued map from a strong f-ring A into a frame L.

We recall from [2] that an f-ring A has a natural topology, its uniform topology, with basic neighbourhoods

$$V_n(a) = \{x \in A : |x - a| < \frac{1}{n}\}, \ \ n = 1, 2, \cdots$$

for each $a \in A$. For any $I, J \in \mathcal{L}(A)$, $\overline{I \cap J} = \overline{I} \cap \overline{J}$. Therefore, $\mathfrak{M}A$, the frame of all closed ℓ-ideals of A, is a sublocal of $\mathcal{L}(A)$ and a frame under the finite meet in $\mathcal{L}(A)$ and the closure of arbitrary joins in $\mathcal{L}(A)$; in particular it is a compact completely regular frame.

Example 4.3. The following are cozero lattice-valued maps:

1. The map $coz_X : C(X) \to \mathcal{O}(X)$ defined by $f \mapsto coz_X(f) = \{x \in X : f(x) \neq 0\}$, where $\mathcal{O}(X)$ is the frame of open subsets of X.
2. The map $coz_L : \mathcal{R}L \longrightarrow L$ defined by $\alpha \mapsto coz_L(\alpha)$, where L is a frame.
3. The map $\iota_A : A \to \mathcal{L}(A)$ given by $x \mapsto [x]$.
4. The map $\bar{\iota}_A : A \to \mathfrak{M}(A)$ defined by $x \mapsto \overline{[x]}$.

Definition 4.4. A lattice-valued map c is called *continuous* if for every $p, q \in \mathbb{Q}$ and $x \in A$,

$$c(\delta^x_{pq}) = \bigvee_{\substack{r,s \in \mathbb{Q}, \\ p<r<s<q}} c(\delta^x_{rs}).$$

Definition 4.5. A lattice-valued map $c \in F(A, L)$ is called *bounded* if

$$\bigvee_{p,q \in \mathbb{Q}} c(\delta^a_{pq}) = \top,$$

for all $a \in A$.

First, we start with the following lemma.

Lemma 4.6. *If c satisfies Condition $(c1)$ of Definition 4.2, then the following are equivalent:*

(1) *The lattice-valued map $c \in F(A, L)$ is a bounded lattice-valued map.*

(2) $\bigvee_{p<q} c(\delta_{pq}^x) = \top$, *for every $x \in A$.*

Proof. (1) ⇒ (2) Since c satisfies Condition ($c1$), we conclude from Lemma 3.3 that if $p \geq q$, then $c(\delta_{pq}^x) = 0$, for every $x \in A$. Therefore,

$$\top = \bigvee_{p,q\in\mathbb{Q}} c(\delta_{pq}^x) = \bigvee_{p<q} c(\delta_{pq}^x),$$

for every $x \in A$.

(2) ⇒ (1) is evident. □

Proposition 4.7. *Let A be a bounded strong f-ring. If $c \in F(A, L)$ satisfies Condition $(c2)$, then c is a bounded lattice-valued map.*

Proof. Suppose $x \in A$. Then there exists $n \in \mathbb{N}$ such that $|x| < n$. Consider $p, q \in \mathbb{Q}$ such that $p < -n - 1$ and $q > 1 + n$. Thus, $-n < x$ implies that $1 < -n - p < x - p$. Consequently, $(x - p)^+ > 1$. Also, $x < n$ implies $1 < q - n < q - x$, and so $(q - x)^+ > 1$. Therefore, $\delta_{pq}^x > 1$ and δ_{pq}^x is invertible. Since c satisfies Condition ($c2$), we conclude that $c(\delta_{pq}^x) = \top$. Therefore,

$$\bigvee_{p,q\in\mathbb{Q}} c(\delta_{pq}^x) = \top,$$

for every $x \in A$. Hence, c is a bounded lattice-valued map. □

The following proposition shows that coz_X and coz_L, mentioned in Examples 4.3(1) and (2), are continuous and bounded.

Proposition 4.8. *Let X and L be a topological space and a frame, respectively. Then*

(1) $coz_X : C(X) \to \mathcal{O}X$ *is a bounded continuous lattice-valued map,*

(2) $coz_L : \mathcal{R}L \to L$ *is a bounded continuous lattice-valued map.*

Proof. (1) Suppose $f \in C(X)$ and $p, q \in \mathbb{Q}$. By Lemma 3.3,

$$\bigvee_{\substack{r,s\in\mathbb{Q},\\ p<r<s<q}} coz_X(\delta^f_{rs}) \subseteq coz_X(\delta^f_{pq}).$$

Now, assume that $x \in c(\delta^f_{pq})$. Then,

$$0 \neq \delta^f_{pq}(x) = \min\{(f(x)-p)^+, (q-f(x))^+\}.$$

If $0 \neq \delta^f_{pq}(x) = (f(x)-p)^+$, then there exists $n \in \mathbb{N}$ such that

$$p < p + \frac{1}{n} < q - \frac{1}{n} < q$$

and

$$x \in coz_X(\delta^f_{(p+\frac{1}{n})(q-\frac{1}{n})}) \subseteq \bigvee_{\substack{r,s\in\mathbb{Q},\\ p<r<s<q}} coz_X(\delta^f_{rs}).$$

If $0 \neq \delta^f_{pq}(x) = (q-f(x))^+$, then a similar proof shows that

$$x \in \bigvee_{\substack{r,s\in\mathbb{Q},\\ p<r<s<q}} coz_X(\delta^f_{rs}).$$

Hence

$$coz_X(\delta^f_{pq}) = \bigvee_{\substack{r,s\in\mathbb{Q},\\ p<r<s<q}} coz_X(\delta^f_{rs}),$$

for every $f \in C(X)$ and for every $p, q \in \mathbb{Q}$, which proves the continuity of coz_X. To prove boundedness, suppose that $f \in C(X)$ and $x \in X$. If $f(x) \notin \mathbb{Z}$, we put $p = \lfloor f(x) \rfloor$ and $q = 2\lfloor f(x)+1 \rfloor$, then we have

$$x \in coz_X(\delta^f_{pq}) \subseteq \bigvee_{\substack{p,q\in\mathbb{Q},\\ p<q}} coz_X(\delta^x_{pq}),$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

If $f(x) \in \mathbb{Z}$, we put $p = f(x) - 1$ and $q = f(x) + 1$, then we have

$$x \in coz_X(\delta^f_{pq}) \subseteq \bigvee_{\substack{p,q\in\mathbb{Q},\\ p<q}} coz_X(\delta^f_{pq}).$$

Therefore,

$$\bigvee_{\substack{p,q\in\mathbb{Q},\\ p<q}} coz_X(\delta^f_{pq}) = X = \top_{\mathcal{P}(X)},$$

for every $f \in C(X)$.

(2) Suppose that $\alpha \in \mathcal{R}L$. Since α is a frame map, we conclude, from [2, Lemma 6], that

$$coz_L(\delta^\alpha_{pq}) = \alpha(p,q) = \bigvee_{\substack{r,s\in\mathbb{Q},\\ p<r<s<q}} \alpha(r,s) = \bigvee_{\substack{r,s\in\mathbb{Q},\\ p<r<s<q}} coz_L(\delta^\alpha_{rs}).$$

Also we have,

$$\bigvee_{p,q\in\mathbb{Q}} coz_L(\delta^\alpha_{pq}) = \bigvee_{p,q\in\mathbb{Q}} \alpha(p,q) = \alpha(\bigvee_{p,q\in\mathbb{Q}} (p,q)) = \alpha(\top) = \top,$$

which shows the continuity and boundedness of coz_L. □

In Proposition 4.10 we study continuity and boundedness of lattice-valued maps ι_A and $\bar{\iota}_A$ mentioned in Example 4.3(3 and 4). First, we need the following lemma.

Lemma 4.9. *Let $p, q \in \mathbb{Q}$ with $p < q$. If $\{e_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Q} \cap (0,\infty)$ such that $r_n = p + e_n < q - e_n = s_n$, then*

$$|\delta^a_{pq} - \delta^a_{r_n s_n}| < 3e_n,$$

for every $a \in A$ and $n \in \mathbb{N}$.

Proof. Since $a \vee s_n - a \vee q \leq 0$ and $(-a) \vee (-r_n) - (-a) \vee (-p) \leq 0$, we conclude that

$$0 \leq (a - s_n)^+ - (a - q)^+ = q - s_n + a \vee s_n - a \vee q \leq e_n$$

and

$$0 \leq (r_n - a)^+ - (p - a)^+ = r_n - p + (-a) \vee (-r_n) - (-a) \vee (-p) \leq e_n,$$

for every $a \in A$ and $n \in \mathbb{N}$. Also, we have

$$\begin{aligned}
&(r_n - a)^+ \vee (a - s_n)^+ - (p-a)^+ \vee (a-q)^+ \\
&= e_n + (p-a) \vee (a-q) \vee (-e_n) \\
&= -(p-a) \vee (a-q) \vee 0 \\
&\leq e_n + (p-a) \vee (a-q) \vee 0 \\
&-(p-a) \vee (a-q) \vee 0 \\
&= e_n
\end{aligned}$$

for every $a \in A$ and $n \in \mathbb{N}$. It is clear that

$$a \wedge b - c \wedge d = (a-c) + (b-d) + c \vee d - a \vee b,$$

for every $a, b, c, d \in A$. Hence

$$\begin{array}{rcl}
|\delta^a_{pq} - \delta^a_{r_n s_n}| & \leq & |(p-a)^+ - (r_n - a)^+| + |(a-q)^+ - (a-s_n)^+| \\
& & +|(r_n - a)^+ \vee (a-s_n)^+ - (p-a)^+ \vee (a-q)^+| \\
& \leq & 3e_n,
\end{array}$$

for every $a \in A$ and $n \in \mathbb{N}$. □

Proposition 4.10. (1) *Let* $A = C((0,1))$ *be the ring of continuous functions on the open interval* $(0,1)$*. Then the lattice-valued map* ι_A *is neither continuous nor bounded.*

(2) *If* A *is a strong* f*-ring then* $\bar{\iota}_A$ *is continuous.*

Proof. (1) Suppose $f : (0,1) \to \mathbb{R}$ is given by $f(x) = x$. If $p, q \in \mathbb{Q}$ and $p \leq q$, then

$$\delta^f_{pq}(x) = \begin{cases} x - p & \text{for } p \leq x \leq \frac{(p+q)}{2}, \\ q - x & \text{for } \frac{(p+q)}{2} \leq x \leq q, \\ 0 & \text{otherwise.} \end{cases}$$

Assume $p, q \in \mathbb{Q}$ and $0 < p \leq q < 1$. We claim that

$$\iota_A(\delta^f_{pq}) \neq \bigvee_{p<r<s<q} \iota_A(\delta^f_{rs}).$$

If $\iota_A(\delta^f_{pq}) = \bigvee_{p<r<s<q} \iota_A(\delta^f_{rs})$, then

$$\begin{array}{rcl}
\delta^f_{pq} \in [\delta^f_{pq}] & = & \bigvee_{p<r<s<q}[\delta^f_{rs}] \\
& = & \sum_{p<r<s<q}[\delta^f_{rs}].
\end{array}$$

Hence, there exists $r_1, s_1, \cdots, r_k, s_k \in \mathbb{Q}$ and $g_1, \ldots, g_k \in A$ such that

$$\delta^f_{pq} = g_1 + \cdots + g_k,$$

$p < r_1, s_1, \ldots, r_k, s_k < q$ and $g_i \in [\delta^f_{r_i s_i}]$ for every $1 \leq i \leq k$. Also, for every $1 \leq i \leq k$, there exists $0 \leq h_i \in A$ such that $|g_i| \leq |\delta^f_{r_i s_i}|h_i$. Therefore,

$$\begin{aligned} |\delta^f_{pq}| &\leq |g_1| + \cdots + |g_k| \\ &\leq |\delta^f_{r_1 s_1}|h_1 + \cdots + |\delta^f_{r_k s_k}|h_k. \end{aligned}$$

Let $r = min\{r_1, \cdots, r_k\}$. Then, if $p < x < r$, we have

$$0 \lneqq x - p = |\delta^f_{pq}|(x) \leq (|\delta^f_{r_1 s_1}|h_1 + \cdots + |\delta^f_{r_k s_k}|h_k)(x) = 0,$$

which is a contradiction. Therefore, ι_A is not a continuous lattice-valued map.

Now suppose that $\iota_A : A \to \mathcal{L}(A)$ is a bounded lattice-valued map. Once again consider $f : (0,1) \to \mathbb{R}$ defined by $f(x) = x$. By the definition of boundedness,

$$f \in A = \bigvee_{\substack{p,q \in \mathbb{Q}, \\ p<q}} \iota_A(\delta^f_{pq}) = \sum_{\substack{p,q \in \mathbb{Q}, \\ p<q}} [\delta^f_{pq}].$$

Hence, there exist $r_1, s_1, \cdots, r_k, s_k \in \mathbb{Q}$ and $g_1, \ldots, g_k \in A$ such that $0 < r_i < s_i < 1$, $f = g_1 + \cdots + g_k$, $g_i \in [\delta^f_{r_i s_i}]$, for every $1 \leq i \leq k$. Hence, for every $1 \leq i \leq k$, there exists $0 \leq h_i \in A$ such that $|g_i| \leq |\delta^f_{r_i s_i}|h_i$. Therefore,

$$\begin{aligned} |f| &\leq |g_1| + \cdots + |g_k| \\ &\leq |\delta^f_{r_1 s_1}|h_1 + \cdots + |\delta^f_{r_k s_k}|h_k. \end{aligned}$$

Let $r = min\{r_1, \cdots, r_k\}$. Then, if $0 < x < r$, we have

$$0 \leq |f|(x) \leq (|\delta^f_{r_1 s_1}|h_1 + \cdots + |\delta^f_{r_k s_k}|h_k)(x) = 0.$$

Hence $\{x \,|\, 0 < x < r\} \subseteq Z(f) = \emptyset$, which is a contradiction. Therefore, $\iota_A : A \to \mathcal{L}(A)$ is not a bounded lattice-valued map.

(2) Suppose that $p, q \in \mathbb{Q}$ with $p < q$ and $a \in A$. By Lemma 3.3,

$$\bigvee_{\substack{r,s \in \mathbb{Q}, \\ p<r<s<q}} \bar{\iota}(\delta^a_{rs}) = \bigvee_{\substack{r,s \in \mathbb{Q}, \\ p<r<s<q}} \overline{[\delta^a_{rs}]} \subseteq \overline{[\delta^a_{pq}]} = \bar{\iota}(\delta^a_{pq}).$$

It is well known that there exists $\{e_n\}_{n \in \mathbb{N}} \subseteq \mathbb{Q} \cap (0, \infty)$ such that $e_n \to 0$ and

$$r_n = p + e_n < q - e_n = s_n.$$

By Lemma 4.9, for every $k \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, if $n \geq k_0$, then

$$|\delta^a_{pq} - \delta^a_{r_n s_n}| \leq 3e_n \leq \frac{1}{k}.$$

Hence

$$V_k(\delta^a_{pq}) \cap \bigvee_{\substack{r,s\in\mathbb{Q},\\ p<r<s<q}} \overline{[\delta^a_{rs}]} \neq \emptyset.$$

Since $\bigvee_{\substack{r,s\in\mathbb{Q},\\ p<r<s<q}} \overline{[\delta^a_{rs}]}$ is a closed ℓ-ideal, we conclude that $\delta^a_{pq} \in \bigvee_{\substack{r,s\in\mathbb{Q},\\ p<r<s<q}} \overline{[\delta^a_{rs}]}$, which follows that

$$\overline{[\delta^a_{pq}]} \subseteq \bigvee_{\substack{r,s\in\mathbb{Q},\\ p<r<s<q}} \overline{[\delta^a_{rs}]}.$$

Therefore, $\bar{\iota_A}$ is a continuous lattice-valued map. □

We close this section with the following question as an open problem:

Does there exist a strong f-ring A such that $\bar{\iota}_A$ is not bounded?

5 $\mathbb{Q}$-compatiblity and coz-compatiblity

In this section, we introduce the notions $\mathbb{Q}$-compatiblity for lattice-valued maps $A \to L$ and correspondingly $\mathbb{Q}$-compatiblity for ring homomorphisms $A \to \mathcal{R}L$.

Definition 5.1. Let $c : A \to L$ be a lattice-valued map. If for every $\diamond \in \{+, \cdot, \vee, \wedge\}$, $a, b \in A$, and $r, s, w, z, p, q \in \mathbb{Q}$,

$$c(\delta^a_{rs}) \wedge c(\delta^b_{wz}) \leq c(\delta^{a\diamond b}_{pq}),$$

whenever $< r, s > \diamond < w, z > \subseteq < p, q >$, then c is called $\mathbb{Q}$-*compatible.*

Definition 5.2. The map $\phi : A \to \mathcal{R}L$ is called *coz-compatible* wherever

$$coz_L(\phi(\delta^a_{pq})) = \phi(a)(p, q),$$

for every $p, q \in \mathbb{Q}$ and $a \in A$.

Let $\phi : A \to \mathcal{R}L$ be a function. We define a map c_ϕ from A into L by $c_\phi(a) = coz_L(\phi(a))$, for all $a \in A$.

The following proposition shows a primary relation between these two notions.

Proposition 5.3. *Let $\psi : A \to \mathcal{R}L$ be an f-ring homomorphism. If $\psi : A \to \mathcal{R}L$ is coz-compatible, then $c_\psi : A \to L$ is $\mathbb{Q}$-compatible.*

Proof. If p, q, r, s, w, and z are rational numbers satisfying $< r, s > \diamond < w, z > \leq < p, q >$, then

$$\begin{aligned} c_\psi(\delta^a_{rs}) \wedge c_\psi(\delta^b_{wz}) &= coz_L(\psi(\delta^a_{rs})) \wedge coz_L(\psi(\delta^b_{wz})) \\ &= \psi(a)(r, s) \wedge \psi(b)(w, z) \\ &\leq \bigvee\{\psi(a)(r, s) \wedge \psi(b)(w, z) : \\ &\qquad < r, s > \diamond < w, z > \leq < p, q >\} \\ &= \psi(a) \diamond \psi(b)(p, q) \\ &= \psi(a \diamond b)(p, q) \\ &= coz_L(\psi(\delta^{(a\diamond b)}_{pq})) \\ &= c_\psi(\delta^{(a\diamond b)}_{pq}). \end{aligned}$$

Therefore, c_ψ is $\mathbb{Q}$-compatible. □

Example 5.4. By giving an example, we discuss the relation between Definitions 5.1, 5.2, and Proposition 5.3. Consider $\mathbb{Q}$ as an f-ring. Define $\theta : \mathbb{Q} \to \mathcal{R}L$, given by $\theta(x) = \mathbf{x}$, which is an f-ring homomorphism. For every $a, p, q \in \mathbb{Q}$, we have

$$coz_L(\theta(\delta^a_{pq})) = coz_L(\delta^{\mathbf{a}}_{\mathbf{pq}}) = \begin{cases} \top & \text{for } p < a < q, \\ \bot & \text{otherwise.} \end{cases}$$

On the other hand,

$$\theta(x)(p, q) = \mathbf{x}(p, q) = \begin{cases} \top & \text{for } p < a < q, \\ \bot & \text{otherwise.} \end{cases}$$

Hence, $coz_L(\theta(\delta^a_{pq})) = \theta(x)(p, q)$. Thus θ is coz-compatible. Now, to calculate $c_\theta : \mathbb{Q} \to L$, we have

$$c_\theta = coz_L(\theta(x)) = coz_L(\mathbf{x}) = \begin{cases} \top & x \neq 0 \\ \bot & x = 0. \end{cases}$$

Finally, note that $c_\theta : \mathbb{Q} \to L$ is $\mathbb{Q}$-compatible, by Proposition 5.3.

In the following proposition, we study $\mathbb{Q}$-compatiblity of items of Example 4.3.

Proposition 5.5. *Let X and L be a topological space and a frame, respectively. Then coz_X and coz_L are $\mathbb{Q}$-compatible.*

Proof. We know that

$$coz_X(\delta^h_{uv}) = \{x \in X \mid u < f(x) < v\},$$

for every $u, v \in \mathbb{Q}$ and $h \in C(X)$. Suppose that $f, g \in C(X)$, $r, s, w, z, p, q \in \mathbb{Q}$ such that $< r, s > \diamond < w, z > \subseteq < p, q >$, where $\diamond \in \{+, \cdot, \vee, \wedge\}$. We have

$$\begin{aligned} x \in coz_X(\delta^f_{rs}) \wedge coz_X(\delta^g_{wz}) &\Rightarrow r < f(x) < s \,\text{and}\, w < g(x) < z \\ &\Rightarrow f(x) \diamond g(x) \in < r, s > \diamond < w, z > \\ &\Rightarrow (f \diamond g)(x) \in < p, q > \\ &\Rightarrow x \in coz_X(\delta^{f\diamond g}_{pq}). \end{aligned}$$

Therefore,

$$coz_X(\delta^f_{rs}) \wedge coz_X(\delta^g_{wz}) \leq coz_X(\delta^{f\diamond g}_{pq}).$$

It proves that coz_X is $\mathbb{Q}$-compatible. But for coz_L, we have

$$\begin{aligned} coz_L(\delta^\alpha_{rs}) \wedge coz_L(\delta^\beta_{wz}) &= \alpha(r, s) \wedge \beta(w, z) \\ &\leq \bigvee\{\alpha(r, s) \wedge \beta(w, z) :< r, s > \diamond < w, z > \subseteq < p, q >\} \\ &= \alpha \diamond \beta(p, q) \\ &= coz_L(\delta^{\alpha\diamond\beta}_{pq}). \end{aligned}$$

So, coz_L is $\mathbb{Q}$-compatible. □

Remark 5.6. About the question that lattice-valued maps ι_A and $\bar{\iota}_A$ are $\mathbb{Q}$-compatible or not, we do not have an exact answer in the general case, but in the case of ring of continuous functions, the answer is positive, which is proved in Proposition 5.7. Any way, we leave the following open question to the readers:

Does there exist a strong f-ring A such that ι_A and $\bar{\iota}_A$ are not $\mathbb{Q}$-compatible?

Proposition 5.7. *Let X be a topological space. Suppose that $A = C(X)$. Then, the lattice-valued maps ι_A and $\bar{\iota}_A$ are $\mathbb{Q}$-compatible.*

Proof. First note that if $coz_X(f) \subseteq coz_X(g)$ then we can conclude that $f \in [g]$. Now let $f, g \in C(X)$. Suppose that $< r, s > \diamond < w, z > \subseteq < p, q >$, where $r, s, w, z, p, q \in \mathbb{Q}$. By $\mathbb{Q}$-compatiblity of coz_X, we have $coz_X(\delta^f_{rs} \wedge \delta^g_{wz}) \subseteq coz_X(\delta^{f\diamond g}_{zw})$, so $\delta^f_{rs} \wedge \delta^g_{wz} \in [\delta^{f\diamond g}_{zw}]$. Therefore,

$$\iota_A(\delta^f_{rs}) \wedge \iota_A(\delta^g_{wz}) = \iota_A(\delta^f_{rs} \wedge \delta^g_{wz}) \subseteq \iota_A(\delta^{f\diamond g}_{zw}),$$

since $\delta^f_{rs}, \delta^g_{wz} \geq 0$. Similarly, $\bar{\iota}_A(\delta^f_{rs}) \wedge \bar{\iota}_A(\delta^g_{wz}) \subseteq \bar{\iota}_A(\delta^{f\diamond g}_{zw})$. □

The partial ordering on $F(A, L)$ is defined by:

$$c_1 \leq c_2 \text{ if and only if } c_1(x) \leq c_2(x) \text{ for all } x \in A.$$

It is clear that $(c_1 \wedge c_2)(x) = c_1(x) \wedge c_2(x)$, for all $x \in A$. Hence $F(A, L)$ is a meet-semilattice.

Proposition 5.8. *Let $c_1, c_2 \in F(A, L)$ be $\mathbb{Q}$-compatible cozero lattice-valued maps. Then $c_1 \wedge c_2$ is a $\mathbb{Q}$-compatible cozero lattice-valued map.*

Proof. If p, q, r, s, w, and z are rational numbers satisfying

$$(r, s) \diamond (w, z) \leq (p, q),$$

then

$$\begin{aligned}(c_1 \wedge c_2)(\delta^a_{rs}) \wedge (c_1 \wedge c_2)(\delta^b_{wz}) &= (c_1(\delta^a_{rs}) \wedge c_2(\delta^a_{rs})) \wedge (c_1(\delta^b_{wz}) \wedge c_2(\delta^b_{wz})) \\ &= (c_1(\delta^a_{rs}) \wedge c_1(\delta^b_{wz})) \wedge (c_2(\delta^a_{rs}) \wedge c_2(\delta^b_{wz})) \\ &\leq c_1(\delta^{a\diamond b}_{pq}) \wedge c_2(\delta^{a\diamond b}_{pq}) \\ &= (c_1 \wedge c_2)(\delta^{a\diamond b}_{pq}).\end{aligned}$$

□

6 Representation of A into $\mathcal{R}L$

As in [1], we recall $\hat{a} : \mathcal{R} \to \mathfrak{M}(A)$ given by $\hat{a}(p,q) = \overline{[(a-p)^+ \wedge (q-a)^+]}$. Then $\hat{a}$ is a frame homomorphism and so it is an element of $\mathcal{R}\mathfrak{M}(A)$. It entails an f-ring homomorphism $\tau_A : A \to \mathcal{R}\mathfrak{M}(A)$ defined by $\tau_A(a) = \hat{a}$. In this section, we consider an arbitrary frame L instead of $\mathfrak{M}(A)$, and introduce a_c and τ_c which are extended to $\hat{a}$ and τ_A, respectively, for each suitable cozero lattice-valued map $c : A \to L$. Also, we study the relations between τ_c and c.

Lemma 6.1. *Let $c \in F(A, L)$ be a bounded continuous cozero lattice-valued map. The following hold*:

(R1) $c(\delta^a_{pq}) \wedge c(\delta^a_{rs}) = c(\delta^a_{(p\vee r)(q\wedge s)})$,

(R2) $c(\delta^a_{pq}) \vee c(\delta^a_{rs}) = c(\delta^a_{ps})$, *whenever* $p \le r < q \le s$,

(R3) $c(\delta^a_{pq}) = \bigvee_{\substack{r,s\in\mathbb{Q},\\ p<r<s<q}} c(\delta^a_{rs})$,

(R4) $\bigvee_{p,q\in Q} c(\delta^a_{pq}) = \top$.

Proof. **(R1)**: Since c satisfies Condition $(c4)$, we conclude, from Lemma 3.3, that

$$\begin{aligned} c(\delta^a_{(p\vee r)(q\wedge s)}) &= c(\delta^a_{pq} \wedge \delta^a_{rs}) \\ &= c(\delta^a_{pq}) \wedge c(\delta^a_{rs}). \end{aligned}$$

for every $p, q, r, s \in \mathbb{Q}$ and $a \in A$.

(R2): Let $p, q, r, s \in \mathbb{Q}$ with $p \le r < q \le s$. Since c satisfies Condition $(c3)$, we conclude, from Lemma 3.3, that

$$\begin{aligned} c(\delta^a_{pq}) \vee c(\delta^a_{rs}) &= c(\delta^a_{pq} \vee \delta^a_{rs}) \\ &= c(\delta^a_{ps}). \end{aligned}$$

(R3) and **(R4)**: They are directly implied from the definitions of continuous and bounded lattice-valued maps. □

Definition 6.2. Define $a_c : \mathcal{R} \to L$ by $a_c(p,q) = c(\delta^a_{pq})$, for every $a \in A$ and $c \in F(A, L)$.

Proposition 6.3. *Let $c \in F(A, L)$ be a bounded continuous cozero lattice-valued map. Then, $r_c = \mathbf{r} \in \mathcal{R}L$, for every $r \in \mathbb{Q}$. In particular, $0_c = \mathbf{0} \in \mathcal{R}L$ and $1_c = \mathbf{1} \in \mathcal{R}L$.*

Proof. Consider $r \in \mathbb{Q}$. Since for every $p, q \in \mathbb{Q}$,

$$r_c(p,q) = \begin{cases} \top & \text{for } p < 0 < q, \\ \bot & \text{otherwise,} \end{cases}$$

we conclude that $r_c = \mathbf{r} \in \mathcal{R}L$. □

Definition 6.4. A cozero lattice-valued map $c : A \to L$ is called *realizable*, if $a_c \in \mathcal{R}L$, for every $a \in A$. Also, the map $c^r : A \to L$ given by $a \mapsto coz_L(a_c)$, is called the *realization* of c.

Proposition 6.5. *A cozero lattice-valued map is realizable if and only if it is bounded and continuous.*

Proof. Note that, in Lemma 6.1, the equations (**R3**) and (**R4**) are properties of being continuous and bounded, respectively. This proves the proposition. □

Remark 6.6. Consider the lattice-valued map coz_L. By Lemma 6 of [2], we have $\alpha_{coz_L}(p,q) = coz_L(\delta^{\alpha}_{pq}) = \alpha(p,q)$. So coz_L is realizable and its realization is itself, that is, $\alpha_{coz_L} = \alpha$, and $coz^r_L(\alpha) = coz_L(\alpha_c) = coz_L(\alpha)$. By Proposition 6.5, the lattice-valued map ι_A is not realizable.

Theorem 6.7. *Let $c \in F(A, L)$ be a bounded continuous cozero lattice-valued map. If c is $\mathbb{Q}$-compatible, then*

$$(a \diamond b)_c = a_c \diamond b_c,$$

for every $a, b \in A$ and $\diamond \in \{+, \cdot, \vee, \wedge\}$.

Proof. Suppose $a, b \in A$, $\diamond \in \{+, \cdot, \vee, \wedge\}$ and $p, q \in \mathbb{Q}$. Since c is $\mathbb{Q}$-compatible,

$$\begin{aligned} a_c \diamond b_c(p,q) &= \bigvee\{a_c(r,s) \wedge b_c(w,z) | \langle r,s\rangle \diamond \langle w,z\rangle \subseteq \langle p,q\rangle\} \\ &= \bigvee\{c(\delta^a_{rs}) \wedge c(\delta^b_{wz}) | \langle r,s\rangle \diamond \langle w,z\rangle \subseteq \langle p,q\rangle\} \\ &\leq \bigvee\{c(\delta^{a\diamond b}_{pq}) | \langle r,s\rangle \diamond \langle w,z\rangle \subseteq \langle p,q\rangle\} \\ &= c(\delta^{a\diamond b}_{pq}) \\ &= (a \diamond b)_c(p,q). \end{aligned}$$

Since a_c, b_c, and $(a\diamond b)_c$ are frame maps and $\mathcal{R}$ is a regular frame, it follows that $a_c \diamond b_c = (a\diamond b)_c$ □

Definition 6.8. Let $c \in F(A, L)$ be a realizable lattice-valued map. We define the map τ_c from A into $\mathcal{R}L$ by $\tau_c(a) = a_c$, for all $a \in A$.

Corollary 6.9. *Let $c \in F(A, L)$ be a bounded continuous cozero lattice-valued map. If c is $\mathbb{Q}$-compatible, then the map τ_c is an f-ring homomorphism and a $\mathbb{Q}$-linear map. Also,*

$$\tau_c(\delta^a_{rs}) = \delta^{\tau_c(a)}_{rs},$$

for every $r, s \in \mathbb{Q}$ and $a \in A$.

Proof. By Proposition 6.3 and Theorem 6.7, we get the result. For the second part, we refer to Proposition 3.2. □

Proposition 6.10. *Let $c \in F(A, L)$ be a bounded continuous cozero lattice-valued map. If c is $\mathbb{Q}$-compatible, then so is c^r.*

Proof. If p, q, r, s, u, and w are rational numbers satisfying $< r, s > \diamond < u, w > \subseteq < p, q >$, then, by Corollary 6.9, we have

$$\begin{aligned}
c^r(\delta^{a\diamond b}_{pq}) &= coz(\delta^{\tau_c(a\diamond b)}_{pq}) \\
&= \tau_c(a\diamond b)(p,q), && \text{by Lemma 6 in [2]} \\
&= (a_c \diamond b_c)(p,q), \\
&= \bigvee\{a_c(r,s)\wedge b_c(u,w) \mid \\
&\qquad < r,s > \diamond < u,w > \subseteq < p,q >\} \\
&\geq a_c(r,s)\wedge b_c(u,w) \\
&= coz(\delta^{a_c}_{rs})\wedge coz(\delta^{b_c}_{uw}), && \text{by Lemma 6 in [2]} \\
&= c^r(\delta^a_{rs})\wedge c^r(\delta^b_{wz}).
\end{aligned}$$

Hence c^r is $\mathbb{Q}$-compatible. □

The following proposition explains the relation between f-ring homomorphisms and $\mathbb{Q}$-linear maps $A \to \mathcal{R}L$ and the bounded continuous $\mathbb{Q}$-compatible cozero lattice-valued maps $A \to L$.

Proposition 6.11. *Let $c \in F(A, L)$ be a bounded continuous $\mathbb{Q}$-compatible cozero lattice-valued map, and $\phi : A \to \mathcal{R}L$ be an f-ring homomorphism $\mathbb{Q}$-linear map. Then, the following hold:*

(1) $c_{\tau_c} = c^r$,

(2) $c^{rr} = c^r$,

(3) $\tau_{c^r} = \tau_c$,

(4) $\tau_{c_\phi} = \phi$.

Proof. (1) is implied from the definition of realization and (2) is implied directly from (1).

(3) By Lemma 6 in [2], for $\alpha \in \mathcal{R}L$ and $p, q \in \mathbb{Q}$,

$$coz_L(\delta^{\alpha}_{pq}) = \alpha(p,q).$$

Then, by Corollary 6.9, we have

$$\begin{aligned}\tau_{c^r}(a)(p,q) &= a_{c^r}(p,q)\\ &= c^r(\delta^a_{pq})\\ &= coz_L(\tau_c(\delta^a_{pq}))\\ &= coz_L(\delta^{\tau_c(a)}_{pq})\\ &= \tau_c(a)(p,q),\end{aligned}$$

for every $a \in A$ and $p, q \in \mathbb{Q}$. Therefore $\tau_{c^r} = \tau_c$.

(4) Let $a \in A$ and $p, q \in \mathbb{Q}$. We have

$$\begin{aligned}a_{c_\phi}(p,q) &= c_\phi(\delta^a_{pq})\\ &= coz_L(\phi(\delta^a_{pq}))\\ &= coz_L(\delta^{\phi(a)}_{pq})\\ &= coz_L((\phi(a)-p)^+ \wedge (q-\phi(a))^+))\\ &= \phi(a)(p,q).\end{aligned}$$

Note that, the last equality holds because for every $\alpha \in \mathcal{R}L$ we have $coz_L((\alpha-p)^+ \wedge (q-\alpha)^+)) = \alpha(p,q)$. Hence, $\tau_{c_\phi}(a) = \phi(a)$, which completes the proof. □

Remark 6.12. Reffering to Bernhard Banaschewski [1], we have a functor $\mathfrak{C}$: **KCRFrm** → **SBFAnn**, from the category of all compact completely regular frames to the category of all strong bounded f-rings, which corresponds every frame L to the ring $\mathfrak{C}(L) = \mathcal{R}L$. The functor $\mathfrak{C}$ has a left adjoint $\mathfrak{M}$ given by $A \mapsto \mathfrak{M}(A)$. There are two appropriate adjunction maps

$$\sigma_L : \mathfrak{M}\mathfrak{C}(L) \to L, \tau_A : A \to \mathfrak{C}\mathfrak{M}(A)$$

given by

$$\sigma_L(\overline{[\alpha]}) = coz_L(\alpha), \tau(a) = \hat{a},$$

where $\hat{a} : \mathcal{R} \to \mathfrak{M}(A)$ is given by

$$\hat{a}(p,q) = \overline{[(a-p)^+ \wedge (q-a)^+]} = \bar{\iota}_A(\delta^a_{pq}).$$

Banaschewski used the adjunction (σ, τ) to describe the pointfree version of Gelfand duality in [1].

Now, according to the methods of this paper, τ_A is fundamentally defined via the cozero map $\bar{\iota}_A : A \to \mathfrak{M}(A)$ and the notation δ^a_{pq}. This fact leads us to define generally $\tau_c : A \to L$ for an arbitrary lattice-valued map $c : A \to L$ instead of the cozero map $\bar{\iota}_A : A \to \mathfrak{M}(A)$, for any frame L (Definition 6.8). The main purpose of this paper is studying the well-defindness of τ_c and its properties corresponding to properties of a lattice-valued map $c : A \to L$.

So, studying τ_c can be a method for thinking about finding a left adjoint for a functor $\mathbf{F}$: **KCRFrm** $\to$ **SBFAnn** (a general functor instead of $\mathfrak{C}$: **KCRFrm** $\to$ **SBFAnn**), that will be a general form of pointfree version of Gelfand duality, which is constructed by B. Banaschewski.

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