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Slimming and regularization of cozero maps

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Dedicated to Professor Bernhard Banaschewski on the occasion of his 90th Birthday

Abstract. Cozero maps are generalized forms of cozero elements. Two particular cases of cozero maps, slim and regular cozero maps, are significant. In this paper we present methods to construct slim and regular cozero maps from a given cozero map. The construction of the slim and the regular cozero map from a cozero map are called slimming and regularization of the cozero map, respectively. Also, we prove that the slimming and regularization create reflector functors, and so we may say that they are the best method of constructing slim and regular cozero maps, in the sense of category theory.

Finally, we give slim regularization for a cozero map $c : M \to L$ in the general case where A is not a $\mathbb{Q}$-algebra. We use the ring and module of fractions, in this construction process.

1 Introduction and preliminaries

The concept of zero sets plays an important role in studying topological spaces X and the ring $C(X)$ [14]. In pointfree topology, the cozero elements

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used in [5–9, 11, 12, 18] are considered as a dual concept of zero sets. Also, for studying why cozero elements take the role of both cozero and zero sets in pointfree topology, see [15].

The main reference of this paper is [17], in which cozero elements were generalized, and this generalized form was called a *cozero map*. In [17] the cozero maps were discussed. Also the slim regular algebraic cozero maps were introduced and represented by the cozero map $c\iota : M \to C\mathcal{L}(M)$, up to isomorphism (Theorem 3.4).

Because of the significance of slim and regular properties for cozero maps, we study the slimming and regularization of cozero maps in this paper.

The necessary background on pointfree topology and f-modules is given in Section 2.

In Section 3, we see some necessary details about the cozero maps, which are mentioned in [17].

Our main topics of this paper start from Section 4. In this section, we present slimming of a cozero map. To this end, we need the quotient of an ℓ-module over an ℓ-ideal, which is prepared in Proposition 4.2. Lemma 4.3 gives the method of constructing a slim cozero map from a given cozero map, and Theorem 4.5 says that this construction method is a reflection in the sense of category theory, and so we call it the *slimming* of a cozero map. A similar construction method for the concept of *regularization* is discussed in Section 5, where Lemma 5.1 and Theorem 5.3 prepare a reflection construction of a regular cozero map from a cozero map. To do this, we need the quotient frame L on a frame congruence $\theta \subseteq L \times L$, which we use in Lemma 5.1. For congruences of frames, see [2, 3]. In Section 6, we give slim regularization for a cozero map $c : M \to L$ in the general case that A is not a $\mathbb{Q}$-algebra. We use the ring and module of fractions, in this construction process. Finally, we give an outlook of slim, regular, and algebraic cozero maps.

2 Background

Here we recall some definitions and results from the literature on frames and ordered algebraic structures. For more details, see the appropriate references given in the paper.

2.1 Pointfree topology [4, 16]: A *frame* is a complete lattice L in which finite meet distribute over arbitrary join. A *frame map* is a lattice morphism preserving arbitrary joins, the top element $\top$, and the bottom $\bot$.

Let L be a frame. We say that a is *rather below* b, and write $a \prec b$, if $a^* \vee b = \top$, where $a^* = \bigvee\{y : y \wedge a = \bot\}$ is the pseudocomplement of a. A frame L is called *regular* if each of its elements is a join of elements rather below it.

An element a of a frame L is said to be *completely below* b, written as $a \prec\prec b$, if there exists a sequence $(c_q)_{q\in[0,1]\cap\mathbb{Q}}$ where $c_0 = a,\ c_1 = b$, and if $p < q$ then $c_p \prec c_q$. A frame L is called *completely regular* if each $a \in L$ is a join of elements completely below it.

An element $a \in L$ is called *compact* if $a = \bigvee S$ implies $a = \bigvee T$ for some finite $T \subseteq S$. A frame L is called *compact* whenever its top element is compact.

2.2 Ordered algebraic structures [4, 10]: An abelian group $(G, +)$ with a partial order $\leq$ is called an *abelian ℓ-group* if $(G, \leq)$ is a lattice, and $a \leq b$ implies $a + c \leq b + c$ for all $a, b, c \in G$.

For an abelian ℓ-group G, and $a, b \in G$, defining $a^+ = a \vee 0,\ \ a^- = (-a) \vee 0,\ \ |a| = a \vee (-a)$ we have $a = a^+ - a^-,\ |a| = a^+ + a^-,\ a^+ \wedge a^- = 0,\ |a + b| \leq |a| + |b|$.

A *partially ordered ring* (*po-ring*) is a ring A with a partial order $\leq$ such that $a \leq b$ and $r \geq 0$ imply $ra \leq rb$ and $a + c \leq b + c$ for all $c \in A$. The po-ring A is called an ℓ-ring if its order is a lattice order.

Let A be a commutative po-ring with an identity 1. A *partially ordered module* (*po-module*) M over A is an A-module with an order $\leq$ such that for every $a, b, c \in M$ and $r \in A$, $a \leq b$ and $r \geq 0$ imply $ra \leq rb$ and $a+c \leq b+c$. The po-module M is called an *ℓ-module* if it is also a lattice.

In particular, if $A = \mathbb{Q}$ is the field of rational numbers with its natural order, then every ℓ-module over A is called a *Riesz space*.

An ℓ-module M is called *Archimedean* if $nx \leq a$ for all $n \in \mathbb{N}$ implies $x \leq 0$.

Suppose that A is an ordered ring and M is an ℓ-module over A. A submodule I of M is called an *ℓ-ideal* if $|x| \leq |a|$ and $a \in I$ imply $x \in I$. The ℓ-module M is called *bounded* if $M = [u]$ for some $u \in M^+$, where $[u]$

is the ℓ-ideal generated by u. Note that if A is an ℓ-ring with identity, A is a bounded ℓ-module over itself.

It is necessary to say that if M is a bounded ℓ-module, we consider a fixed element $u \in M$ such that $M = [u]$. In this case, denote $p.u$ by $\mathbf{p}$ for all $p \in \mathbb{Q}$. In the particular case of $M = A$, consider the fixed element $u = 1$, and hence $\mathbf{p} = p.1$, where 1 is the identity of A.

The set of all ℓ-ideals of M is denoted by $\mathcal{L}(M)$. We have:

Let A be an ordered ring and M be an ℓ-module over A. Then $\mathcal{L}(M)$ is a frame. Moreover, $\mathcal{L}(M)$ is compact if and only if M is bounded (Theorem 2.1 of [10]).

An ℓ-module M over an ordered ring A is called an *f-module* if for every $a \in A^+$ and $m, n \in M^+$, $a(m \wedge n) = am \wedge an$.

Let A be a $\mathbb{Q}$-algebra ordered ring. For a bounded ℓ-module M over A the neighbourhoods

$$V_n(a) = \{x \in M : |x - a| < \frac{\mathbf{1}}{\mathbf{n}}\},$$

for each $a \in M$ and $n \in \mathbb{N}$ determine a uniform topology on M with $\{V_n(a) : a \in A, n \in \mathbb{N}\}$ as its basis. Moreover, we have that the operations $\diamond : M^2 \to M$ are uniformly continuous, where $\diamond \in \{+, \wedge, \vee\}$. Furthermore, if A is a bounded ℓ-ring then the scalar multiplication $A \times M \to M$ is continuous.

Suppose that A is an ordered ring and M is an ℓ-module over A. Then the map $c_M : \mathcal{L}(M) \to \mathcal{L}(M)$ given by $c_M(I) = \overline{I}$, where $\overline{I}$ is the closure of I under the uniform topology, is proved to be a nucleus and $Fix(c_M)$ consisting of all closed ℓ-ideals of M is denoted by $C\mathcal{L}(M)$.

For any bounded f-module M over a bounded strong ordered ring A, $C\mathcal{L}(M)$ is completely regular.

Let A be an ordered ring and S be a multiplicative subset of A containing 1. Consider the ring $S^{-1}A$ of fractions. If $S \subseteq A^+$, we define an order on $S^{-1}A$ by $a/s \leq b/t$ if and only if there exists $w \in S$ such that $w(ta - sb) \leq 0$, which makes $S^{-1}A$ an ordered ring. If A is an f-ring, then so is $S^{-1}A$ with $a/s \diamond b/t = (ta \diamond sb)/st$ for each $\diamond \in \{+, \wedge, \vee\}$ and $|a/s| = |a|/s$. Suppose that M is an f-module over the ordered ring A and S is a multiplicative subset of A with $S \subseteq A^+$. Then $S^{-1}M$ can be ordered as above. It is proved that if A is an ordered ring and M is an f-module then so is $S^{-1}M$. Moreover, $a/s \diamond b/t = (ta \diamond sb)/st$ for $\diamond \in \{+, \wedge, \vee\}$ and also $|a/s| = |a|/s$.

2.3 Cozero elements: Recall from [5] that the frame $\mathcal{R}$ of reals is obtained by taking the ordered pairs (p, q) of rational numbers as generators and imposing the relations

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,

(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$,

(R3) $(p, q) = \bigvee\{(r, s)|p < r < s < q\}$,

(R4) $\top = \bigvee\{(p, q)|$ all $p, q\}$.

Note that the pairs (p, q) in $\mathcal{R}$ and the open intervals $(p, q) = \{x \in \mathbb{R} : p < x < q\}$ in the frame $O\mathbb{R}$ of open sets of the space $\mathbb{R}$ (with the usual topology), have the same role; in fact there is a frame isomorphism $\lambda : \mathcal{R} \to O\mathbb{R}$ such that $\lambda(p, q) = (p, q)$.

The set $C(L)$ of all frame homomorphisms from $\mathcal{R}$ to L has been studied as an f-ring by B. Banaschewski [5].

Corresponding to every continuous operation $\diamond : \mathbb{R}^2 \to \mathbb{R}$ (in particular $+, \cdot, \wedge, \vee$) we have an operation on $C(L)$, denoted by the same symbol $\diamond$, defined by

$$\alpha \diamond \beta(p, q) = \bigvee\{\alpha(r, s) \wedge \beta(z, w)(r, s) \diamond (z, w) \leq (p, q)\},$$

where $(r, s) \diamond (z, w) \leq (p, q)$ means that for each $r < x < s$ and $z < y < w$ we have $p < x \diamond y < q$. For every $r \in \mathbb{R}$, define the constant frame map $\mathbf{r} \in C(L)$ by $\mathbf{r}(p, q) = \top$, whenever $p < r < q$, and otherwise $\mathbf{r}(p, q) = \bot$.

$C(L)$ is also a Riesz space with the scalar multiplication $r\alpha = (r1).\alpha$, where $r \in \mathbb{Q}$, $\alpha \in C(L)$ and $\cdot$ is the ring multiplication of $C(L)$ (see [13]).

For each $\alpha \in C(L)$ and $A \subseteq C(L)$, let $coz(\alpha) = \alpha(-, 0) \vee \alpha(0, -)$ and $coz(A) = \{coz(\alpha) : \alpha \in A\}$. For any $\alpha, \beta \in C(L)$ we have

- $coz(\mathbf{0}) = \bot$ and $coz(\mathbf{1}) = \top$.
- $coz(\alpha + \beta) \leq coz(\alpha) \vee coz(\beta)$, and if $\alpha, \beta \geq 0$, the equality holds.
- $coz(|\alpha|) = coz(\alpha)$.
- $coz(\alpha\beta) = coz(\alpha) \wedge coz(\beta)$.
- If $\alpha, \beta \geq 0$ then $coz(\alpha \wedge \beta) = coz(\alpha) \wedge coz(\beta)$.
- For every $\alpha \geq 0$, $coz(\alpha) = \bigvee\{coz(\alpha - \mathbf{p})^+|0 < p \leq 1\}$.

3 Cozero maps

In this section, we bring some necessary definitions, examples, and results about cozero maps (see [17] for more details).

Definition 3.1. Suppose M is an ℓ-module over a ring A and L is a frame. A map $c : M \to L$ is said to be a *cozero map* if for every $x, y \in M$ and $a \in A$

(C1) $c(x+y) \leq c(x) \vee c(y)$ and $c(ax) \leq c(x)$.

(C2) $c(|x|) = c(x)$.

(C3) For every $x, y \geq 0$, $c(x \wedge y) = c(x) \wedge c(y)$ and $c(x+y) = c(x) \vee c(y)$.

(C4) If $M = [u]$, $c(u) = \top$.

(C5) $c(0) = \bot$.

A cozero map c is called

slim if

(SC) $c(x) = \top$ implies $x = 0$.

algebraic if

(AC) L is generated by $\{c(a) : a \in M\}$.

regular (whenever M is bounded and A is a $\mathbb{Q}$-algebra) if

(RC) For every $a \in M^+$,

$$c(a) = \bigvee \{c((a - \mathbf{p})^+) : 0 < p \leq 1\}.$$

Here, we exhibit some important examples of cozero maps.

Example 3.2. (1) Let X be a topological space. Then, $coz_X : C(X) \to O(X)$ given by $\alpha \mapsto coz(\alpha) = \{x \in X : \alpha(x) \neq 0\}$ is the classic example of a cozero map. In the general case, the map $coz_L : C(L) \to L$ given by $\alpha \mapsto coz(\alpha) = \alpha(-,0) \vee \alpha(0,-)$ is the pointfree version of the classic example. Note that coz_L is slim regular for every frame L and it is algebraic if and only if L is completely regular (by [5], page 44, Corollary 2).

(2) Suppose that M is an ℓ-module. Let $\iota_M : M \to \mathcal{L}(M)$ be given by $\iota_M(x) = [x]$, where $[x]$ is the ℓ-ideal generated by x. Then, ι_M is a slim cozero map because

$$[\mathbf{1}] = M = \top, [x] = [\mathbf{0}] = \bot \Leftrightarrow x = \mathbf{0}, [x+y] \subseteq [x] \vee [y], [ax] \subseteq [x],$$

and for every $x, y \geq 0$,

$$[x+y] = [x \vee y] = [x] \vee [y], [x \wedge y] = [x] \wedge [y].$$

Moreover, ι_M is algebraic. The regularity of ι_M is equivalent to complete regularity of $\mathcal{L}(M)$ (see [17]).

(3) Suppose that M is an ℓ-module. Then, the map $c\iota_M : M \to C\mathcal{L}(M)$ is given by $c\iota_M(a) = \langle a \rangle$, where $\langle a \rangle$ is the closure of $[a]$. The map $c\iota_M$ is a cozero map because

$$\langle \mathbf{1} \rangle = \top, \langle \mathbf{0} \rangle = \bot, \langle x + y \rangle \subseteq \langle x \rangle \vee \langle y \rangle, \langle ax \rangle \subseteq \langle x \rangle,$$

and for every $x, y \geq 0$,

$$\langle x + y \rangle = \langle x \rangle \vee \langle y \rangle, \langle x \wedge y \rangle = \langle x \rangle \wedge \langle y \rangle.$$

Note that $c\iota_M$ is slim if and only if M is Archimedean. Also, $c\iota_M$ is algebraic, and regular.

The following theorems show that the regularity of cozero maps is related to complete regularity of their codomains.

Theorem 3.3. [17] *Suppose that M is a bounded f-module over a $\mathbb{Q}$-algebra ordered ring. If $c : M \to L$ is regular and algebraic, then L is completely regular.*

Theorem 3.4. [17] *Suppose that M is a bounded f-module, A is a $\mathbb{Q}$-algebra and $c : M \to L$ is a cozero map. If c is regular, then there is a frame map $\sigma : C\mathcal{L}(M) \to L$ given by $\sigma(I) = \bigvee c[I]$ such that $\sigma \circ c\iota_M = c$. Moreover, if L is compact and c is slim and algebraic then σ is an isomorphism.*

4 Slimming of cozero maps

In this section we present a method to construct a slim cozero map from any given cozero map. For a given cozero map $c : M \to L$, let $I_c = \{x \in M : c(x) = \bot\}$. Then we define the cozero map $\mathbf{s}c : M/I_c \to L$, by $\mathbf{s}c(x + I_c) = c(x)$, as the slimming of c. Before that, we discuss about the quotient of an ℓ-module over an ℓ-ideal.

Definition 4.1. Let A be an ordered ring, M be an ordered module over A, and I be a submodule of M. Define an order on $\frac{M}{I}$ by $x + I \leq y + I$ if $y - x = m + u$ for some $m \in M^+$ and $u \in I$.

Proposition 4.2. *The following statements hold*:

(1) $x+I \leq y+I$ *if and only if* $y-x \geq m+u$ *for some* $m \in M^+$ *and* $u \in I$.

(2) *If* I *is a convex submodule, then* $\frac{M}{I}$ *is an ordered module over* A.

(3) *If* M *is an* ℓ*-module and* I *is* ℓ*-ideal, then* $\frac{M}{I}$ *is an* ℓ*-module.*

(4) *For every* $m \in M$, $|m+I| = |m|+I, (m+I)^+ = m^+ + I$, *and* $(m+I)^- = m^- + I$.

(5) *If* M *is an* f*-module, then so is* $\frac{M}{I}$.

(6) *If* M *is bounded, then so is* $\frac{M}{I}$.

Proof. (1) Suppose that $y-x \geq m+u$ and let $m_0 = y-x-u$. Then we have $m_0 \geq m \geq 0$ and $y-x = m_0+u$, where $m_0 \in M^+$ and $u \in I$.

(2) Reflexivity and transitivity of $\leq$ are clear. To show the symmetry, let $x, y \in M$ be such that $x+I \leq y+I$ and $x+I \geq y+I$. Then, there are $m, n \in M^+$ and $u, v \in I$ such that $y-x = m+u$ and $x-y = n+v$. Hence $0 = m+n+u+v$. Thus $m+n = -u-v \in I$. Since I is convex and $0 \leq m \leq m+n \in I$, $m \in I$. Hence $y-x = m+u \in I$, that is $x+I = y+I$. Therefore, $(\frac{M}{I}, \leq)$ is a partially ordered set. To check that $\frac{M}{I}$ is an ordered module over A, let $x, y, z \in M$ be such that $x+I \leq y+I$. Then for $a \in A^+$, we have $y-x = m+u$, where $m \in M^+$ and $u \in I$. So $a(y-x) = am+au$. Hence $a(x+I) \leq a(y+I)$ and $(x+I)+(z+I) = (x+z)+I \leq (y+z)+I = (y+I)+(z+I)$. So (2) is proved.

To show (3), it is enough to show that $(\frac{M}{I}, \leq)$ is a lattice. For this, we prove $(x+I) \vee (y+I) = x \vee y + I$ and $(x+I) \wedge (y+I) = x \wedge y + I$. Since $x \vee y \geq x, y$, we have $x+I, y+I \leq x \vee y + I$. Now, let $m \in M$ be such that $x+I, y+I \leq m+I$. We prove that $x \vee y + I \leq m+I$. We have $m-x = m_0+u, m-y = n_0+v$ for some $m_0, n_0 \in M^+$ and $u, v \in I$. Hence

$$\begin{aligned}
m - x \vee y &= (m-x) \wedge (m-y) \\
&= (m_0+u) \wedge (n_0+v) \\
&= (m_0+u-v) \wedge n_0 + v \\
&= (m_0+u_1) \wedge n_0 + v, u_1 = u-v \in I \\
&\geq (m_0+u_1) \wedge (n_0+u_1) \wedge m_0 \wedge n_0 + v \\
&= (m_0 \wedge n_0 + u_1) \wedge m_0 \wedge n_0 + v \\
&= m_0 \wedge n_0 + u_1 \wedge 0 + v \\
&= c_0 + w,
\end{aligned}$$

where $c_0 = m_0 \wedge n_0 \geq 0$, and $w = u_1 \wedge 0 + v \in I$. By (1), $m+I \geq x \vee y + I$, and it proves the equation $(x+I)\vee(y+I)=x\vee y+I$. For the other equation, note that $x\wedge y=-((-x)\vee(-y))$.

It is obvious that (3) implies (4) and (5).

To show (6), let M be a bounded ℓ-module and $M=[u]$ for some $u\in M^+$. We prove that $\frac{M}{I}=[u+I]$. Let $m\in M^+$. Then there is $a\in A^+$ such that $m\leq au$. So, $m+I\leq a(u+I)$. Therefore $\frac{M}{I}$ is a bounded ℓ-module. This completes the proof. □

Lemma 4.3. *Suppose that $c:M\to L$ is a cozero map. The following statements hold*:

(1) *The set $I_c=\{x\in M : c(x)=\bot\}$ is an ℓ-ideal of M.*

(2) *The map $\mathrm{s}c: M/I_c\to L$ given by $\mathrm{s}c(x+I_c)=c(x)$ is a well-defined slim cozero map.*

(3) *We have $\mathrm{s}c\circ\gamma_c=c$, where $\gamma_c: M\to M/I_c$ is the natural quotient map.*

(4) *The cozero map $\mathrm{s}c$ has the following property*:

For every slim cozero map $\kappa: M_1\to L$ and $f: M\to M_1$, with $\kappa\circ f=c$, there exists a unique ℓ-homomorphism $\overline{f}: M/I_c\to M_1$ such that $\overline{f}\circ\gamma_c=f$ and $\kappa\circ\overline{f}=\mathrm{s}c$.

Proof. (1) Since for every $x,y\in I_c$ and $a\in I$, $c(x+y)\leq c(x)\vee c(y)=\bot$ and $c(ax)\leq c(x)=\bot$, we have I_c is a submodule of M. Suppose that $|x|\leq|y|$ and $y\in I_c$. Hence $c(x)=c(|x|)\leq c(|y|)=c(y)=\bot$, and so $x\in I_c$. That is, I_c is an ℓ-ideal.

(2) To show that $\mathrm{s}c$ is well-defined, assume that $x+I_c=y+I_c$. Hence $c(x-y)=\bot$. Thus

$$c(x)=c(|x|)\leq c(|x-y|+|y|)=c(|x-y|)\vee c(|y|)=c(x-y)\vee c(y)=c(y).$$

Similarly $c(y)\leq c(x)$. So $\mathrm{s}c$ is well defined. It is easy to check the properties (C1)-(C5) of the definition of a cozero map for $\mathrm{s}c$. Also, $\mathrm{s}c$ is obviously slim.

(3) is clear by the definition of $\mathrm{s}c$.

(4) Define $\overline{f}: M/I_c\to M_1$ by $\overline{f}(x+I_c)=f(x)$. Since κ is slim, $I_c\subseteq\ker f$. So $\overline{f}$ is a well-defined map. By the definition of the operations on M/I_c, since f is an ℓ-homomorphism, so is $\overline{f}$. It is clear that $\overline{f}\circ\gamma_c=f$. The uniqueness of f is clear by the equation $\overline{f}\circ\gamma_c=f$. Finally, we prove the equation $\kappa\circ\overline{f}=c$. For this,

$$\kappa\circ\overline{f}\circ\gamma_c=\kappa\circ f=c=\mathrm{s}c\circ\gamma_c,$$

since γ_c is onto and $\kappa \circ \overline{f} = \mathbf{s}c$. This proves (4). □

Definition 4.4. The category of all ℓ-rings (commutative with identity), with ℓ-homomorphisms between them, is denoted by $\ell\mathcal{R}\mathbf{ng}$.

Let A be an ordered ring (commutative with identity). The category of all ℓ-modules over A, with ℓ-module homomorphisms between them, is denoted by $\ell\mathcal{M}\mathbf{od}(\mathbf{A})$. Note that if $A = \mathbb{R}$, then $\ell\mathcal{M}\mathbf{od}(\mathbf{A}) = \mathbf{Rsz}$ is the category of all Riesz spaces with Riesz maps. Let $\mathcal{X}$ be a subcategory of $\ell\mathcal{R}\mathbf{ng}$ or $\ell\mathcal{M}\mathbf{od}(\mathbf{A})$.

Here we introduce a category denoted by $\mathcal{X}\mathbf{Coz}$, whose objects are all cozero maps $c : X \to L$ from an object of $\mathcal{X}$ into some frame. To introduce the morphisms, let X_1 and X_2 be two objects of $\mathcal{X}$. Suppose $c_1 : X_1 \to L_1$ and $c_2 : X_2 \to L_2$ are two cozero maps. A morphism from c_1 to c_2 is a pair (α, f) such that $\alpha : X_1 \to X_2$ is a morphism in $\mathcal{X}$ and $f : L_1 \to L_2$ is a frame map, such that the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{c_1} & L_1 \\ \alpha\big\downarrow & & \big\downarrow f \\ X_2 & \xrightarrow{c_2} & L_2 \end{array}$$

commutes. The full subcategory of $\mathcal{X}\mathbf{Coz}$ consisting of all slim cozero maps is denoted by $\mathcal{X}\mathbf{SCoz}$.

We have an assignment $S : \mathcal{X}\mathbf{Coz} \to \mathcal{X}\mathbf{SCoz}$ given by $c \mapsto \mathbf{s}c$ and $(\alpha, f) \mapsto (\gamma_{c_2} \circ \overline{\alpha}, f)$, where $\overline{\alpha} \circ \gamma_{c_1} = \alpha$, for the given objects $c_i : X_i \to L_i$ (i=1,2).

Theorem 4.5. *Let $\mathcal{X}$ be a subcategory of $\ell\mathcal{R}\mathbf{ng}$ or $\ell\mathcal{M}\mathbf{od}(\mathbf{A})$, which is closed under quotients. Then $\mathcal{X}\mathbf{SCoz}$ is a reflective subcategory of $\mathcal{X}\mathbf{Coz}$, and $S : \mathcal{X}\mathbf{Coz} \to \mathcal{X}\mathbf{SCoz}$ is the reflector. Moreover $s_c = (\gamma_c, id_L) : c \to \mathbf{s}c$ is the reflection morphism.*

Proof. Using the notations in Lemma 4.3, consider the morphism $s_c = (\gamma_c, id_L) : c \to \mathbf{s}c$, where $c : M \to L$ and $\mathbf{s}c : M/I_c \to L$. Since $\mathcal{X}$ is closed under quotients, M/I_c is an object of $\mathcal{X}$. Now, we prove that s_c has the universal property; that is, for every $t = (\alpha, f) : c \to \kappa$ such that $\kappa : M' \to L'$ is slim, there is a unique morphism $\overline{t} : \mathbf{s}c \to \kappa$ with $\overline{t} \circ s_c = t$.

First we show that $I_c \subseteq \ker \alpha$. Let $x \in I_c$. Hence $c(x) = \bot$, and so $f(c(x)) = \bot$. Since $t : c \to \kappa$ is a morphism, $f \circ c = \kappa \circ \alpha$. Hence

$\kappa(\alpha(x)) = f(c(x)) = \perp$, and since κ is slim, $\alpha(x) = 0$. So, $I_c \subseteq \ker\alpha$. Therefore the map $\overline{\alpha} : M/I_c \to M'$, given by $\overline{\alpha}(x + I_c) = \alpha(x)$, is a well-defined ℓ-homomorphism. Now, let $\bar{t} = (\overline{\alpha}, f) : \mathbf{s}c \to \kappa$. Then $\bar{t}$ is a morphism, because

$$(\kappa \circ \overline{\alpha}) \circ \gamma_c = \kappa \circ (\overline{\alpha} \circ \gamma_c) = \kappa\alpha = f \circ c = f \circ (\mathbf{s}c \circ \gamma_c) = (f \circ \mathbf{s}c) \circ \gamma_c.$$

Since γ_c is onto, $\kappa \circ \overline{\alpha} = f \circ \mathbf{s}c$. Also we have

$$\bar{t} \circ s_c = (\overline{\alpha}, f) \circ (\gamma_c, id_L) = (\overline{\alpha} \circ \gamma_c, f) = (\alpha, f) = t.$$

To show the uniqueness of $\bar{t}$ with this property, suppose that $s = (\beta, g) : \mathbf{s}c \to \kappa$ has the same property. Hence

$$(\alpha, f) = t = \bar{s} \circ s_c = (\beta, g) \circ (\gamma_c, id_L) = (\beta \circ \gamma_c, g),$$

and so $(\alpha, f) = (\beta \circ \gamma_c, g)$. Thus $f = g$ and $\alpha = \beta \circ \gamma_c$. Since $\alpha = \overline{\alpha} \circ \gamma_c$, we have $\beta \circ \gamma_c = \overline{\alpha} \circ \gamma_c$. Thus $\beta = \overline{\alpha}$. So $s = (\beta, g) = (\overline{\alpha}, f) = \bar{t}$. □

5 Regularization

Let M be a bounded ℓ-module over A and $c : M \to L$ be a cozero map. We want to construct a regular cozero map from c. Let Θ be the congruence generated by $\{(c(a), \bigvee_{0<p\leq 1} c(a - \mathbf{p})^+) : a \in M\}$, and let $L_c = L/\Theta$, $q_c = \gamma_\Theta$ be the natural quotient and the natural map, respectively (for more information on frame congruences, see [2, 3]). Consider $q_c \circ c : M \to L_c$. It is clear that $q_c \circ c$ is a cozero map. Because of Lemma 5.1, we say $q_c \circ c$ is the *regularization* of c.

Lemma 5.1. *Let M be a bounded ℓ-module over a $\mathbb{Q}$-algebra A and $c : M \to L$ be a cozero map. Then $q_c \circ c : M \to L_c$ is a regular cozero map. Moreover, for every frame map $f : L \to L'$ such that $f \circ c$ is regular, there is a unique frame map $\tilde{f} : L_c \to L'$ with $\tilde{f} \circ q_c = f$. Also, if f is onto, then so is $\tilde{f}$.*

Proof. First note that q_c is an onto frame map and $q_c \circ c : M \to L_c$ is a regular cozero map, since for every $a \in M$,

$$q_c(c(a)) = q_c(\bigvee_{0<p\leq 1} c(a - \mathbf{p})^+) = \bigvee_{0<p\leq 1} q_c(c(a - \mathbf{p})^+).$$

Thus $q_c \circ c$ is regular. Now, let $f : L \to L'$ be a frame map such that $f \circ c$ is regular. Hence, for every $a \in M$ we have

$$f \circ c(a) = \bigvee_{0<p\leq 1} f \circ c(a - \mathbf{p})^+ = f(\bigvee_{0<p\leq 1} c(a - \mathbf{p})^+),$$

and so,

$$(c(a), \bigvee_{0<p\leq 1} c(a - \mathbf{p})^+) \in \ker f.$$

Therefore $\Theta \subseteq \ker f$. Define $\tilde{f} : L_c \to L'$ by $\tilde{f}(x/\Theta) = f(x)$. Since $\Theta \subseteq \ker f$, $\tilde{f}$ is well-defined.

By the definition of frame quotients, it is clear that $\tilde{f}$ is a frame map. By the definition of $\tilde{f}$, we have $\tilde{f} \circ q_c = f$ and it is unique, since q_c is onto. Finally, $\tilde{f}$ is obviously onto and the proof is complete. □

Definition 5.2. The category of all $\mathbb{Q}$-algebra ℓ-rings together with ℓ-homomorphisms between them, preserving identity, is denoted by $\mathbb{Q}\ell\mathcal{R}\mathbf{ng}$.

Let A be a $\mathbb{Q}$-algebra ordered ring (commutative with identity). The category of all bounded ℓ-modules over A, with bounded ℓ-module homomorphisms (preserving the identity) between them is denoted by $\mathbf{B}\ell\mathcal{M}\mathbf{od}(\mathbf{A})$.

Let $\mathcal{X}$ be a subcategory of $\mathbb{Q}\ell\mathcal{R}\mathbf{ng}$ or $\mathbf{B}\ell\mathcal{M}\mathbf{od}(\mathbf{A})$. The full subcategory of all regular cozero maps $X \to L, X \in \mathcal{X}$, is denoted by $\mathcal{X}\mathbf{RCoz}$.

Now, we introduce an assignment $\mathbf{R} : \mathcal{X}\mathbf{Coz} \to \mathcal{X}\mathbf{RCoz}$ given by $\mathbf{R}c = c \to q_c \circ c$. Let $(\alpha, f) : c \to c'$ be a morphism in $\mathcal{X}\mathbf{Coz}$, where $c : M \to L$ and $c' : M' \to L'$. Define $\mathbf{R}(\alpha, f) = (\alpha, q_{c'} \circ \tilde{f})$, where $\tilde{f} : L_c \to L'$ is the unique frame map introduced in Lemma 5.1.

Theorem 5.3. *Let $\mathcal{X}$ be a subcategory of $\mathbb{Q}\ell\mathcal{R}\mathbf{ng}$ or $\mathbf{B}\ell\mathcal{M}\mathbf{od}(\mathbf{A})$. The category $\mathcal{X}\mathbf{RCoz}$ is a reflective subcategory of $\mathcal{X}\mathbf{Coz}$. Moreover, $r_c : c \to q_c \circ c = \mathbf{R}c$ is the reflection morphism, and $\mathbf{R} : \mathcal{X}\mathbf{Coz} \to \mathcal{X}\mathbf{RCoz}$ is the reflector.*

Proof. It is enough to show that for every cozero map $c : M \to L$, the morphism $r_c : c \to \mathbf{R}c$ has the universal property. For this, let $\mu = (\alpha, f) : c \to c'$ be a morphism in $\mathcal{X}\mathbf{Coz}$ such that $c' : M' \to L'$ is regular.

Since $\alpha \in \mathcal{X}$, α is a bounded ℓ-homomorphism. Now, we show that $c' \circ \alpha$ is regular. Let $t \in M$. Then

$$\begin{aligned} c' \circ \alpha(t) = c'(\alpha(t)) &= \bigvee\{c'((\alpha(t) - \mathbf{p})^+) : 0 < p \leq 1\} \\ &= \bigvee\{c'(\alpha((t - \mathbf{p})^+)) : 0 < p \leq 1\} \\ &= \bigvee\{c' \circ \alpha((t - \mathbf{p})^+) : 0 < p \leq 1\}. \end{aligned}$$

So, $c' \circ \alpha$ is a regular cozero map. Since $f \circ c = c' \circ \alpha$, $f \circ c$ is regular too, and by Lemma 5.1, there is a unique $\tilde{f} : L_c \to L'$ such that $\tilde{f} \circ q_c = f$. Let $\tilde{\mu} = (\alpha, \tilde{f}) : \mathbf{R}c \to c'$. We have

$$\tilde{\mu} \circ r_c = (\alpha, \tilde{f}) \circ (id_M, q_c) = (\alpha, \tilde{f} \circ q_c) = (\alpha, f) = \mu.$$

Also, the uniqueness of $\tilde{\mu}$ is implied from the uniqueness of $\tilde{f}$. So, r_c has the universal property. □

Remark 5.4. Let M be a bounded ℓ-module over a $\mathbb{Q}$-algebra A. It is obvious that for a cozero map $c : M \to L$, $\mathbf{s}c = c$ if and only if c is slim, and $\mathbf{R}c = c$ if and only if c is regular. In particular, $\mathbf{s}^2 = \mathbf{s}$ and $\mathbf{R}^2 = \mathbf{R}$.

Note that "slimming of regularization" and "regularization of slimming" are not the same, that is $\mathbf{s} \circ \mathbf{R} \neq \mathbf{R} \circ \mathbf{s}$. More accurately, the regularization of the slimming of a cozero map is not slim in general, but the slimming of the regularization of any cozero map is both regular and slim. In fact, Example 5.5 below gives an example of a slim cozero map such that its regularization is not slim. But Lemma 5.6 below shows that the slimming of a regular cozero map is regular.

In other words, to obtain a slim and regular cozero map from a cozero map, the functor $\mathbf{s} \circ \mathbf{R}$ is the right one. That is, first we must obtain its regularization and then its slimming. In a more precise term, let M be a bounded ℓ-module over a $\mathbb{Q}$-algebra ordered ring A. To obtain the slim regularization of $c : M \to L$, let $I_c^* = \{x \in M : q_c(c(x)) = \bot\}$. Hence I_c^* is an ℓ-ideal containing I_c. Consider the cozero map $c^* : M/I_c^* \to L_c$, where $c^* \circ \gamma = q_c \circ c$. We have that c^* is the slim regularization of c.

Example 5.5. Consider the cozero map $\kappa = \iota_M : M \to \mathcal{L}(M)$ given by $\kappa(x) = [x]$, where $[x]$ is the ℓ-ideal generated by x. By Example 3.2(2), κ is slim, but it is not regular. Consider $q_\kappa \circ \kappa$, the regularization of κ.

Let M be a non-Archimedean bounded f-module over a $\mathbb{Q}$-algebra ordered ring A. Then, $\overline{[0]} \neq [0]$ (because of Example 3.2(3)). Let $0 \neq x \in \overline{[0]}$. By Lemma 5.7(1), $q_\kappa \circ \kappa(x) = q_\kappa([x]) \leq q_\kappa(\overline{[0]}) = q_\kappa([0]) = 0$. So the cozero map $q_\kappa \circ \kappa = \mathbf{R}(\kappa)$ is not slim.

Lemma 5.6. *If $c : M \to L$ is a regular cozero map, then* $\mathbf{s}c$ *is regular.*

Proof. Let $c : M \to L$ be a regular cozero map. Suppose $\overline{x} = x + I_c$. Thus

$$\mathbf{s}c(\overline{x}) = c(x) = \bigvee_{0<p\leq 1} c(x-\mathbf{p})^+ = \bigvee_{0<p\leq 1} \mathbf{s}c\overline{(x-\mathbf{p})^+} = \bigvee_{0<p\leq 1} \mathbf{s}c(\overline{x}-\mathbf{p})^+.$$

So, $\mathbf{s}c$ is regular. □

Lemma 5.7. *Suppose that M is a bounded f-module over a $\mathbb{Q}$-algebra A. Consider the cozero map $\kappa = \iota_M : M \to \mathcal{L}(M)$. We have*

(1) *For every $I \in \mathcal{L}(M)$, $q_\kappa(\overline{I}) = q_\kappa(I)$,*

(2) *$\widetilde{c_M} : (\mathcal{L}(M))_\kappa \to C\mathcal{L}(M)$ is an isomorphism such that $\widetilde{c_M} \circ q_\kappa = c_M$.*

Proof. (1) Let $x \in \overline{I}$ and $0 < p \leq 1$. Then there exists $a \in I$ such that $|x - a| < \mathbf{p}$. Hence $x - a < \mathbf{p}$. So $x - \mathbf{p} < a$, and thus $0 \leq (x-\mathbf{p})^+ \leq a^+$. Since $a^+ \in I$, $(x-\mathbf{p})^+ \in I$. Therefore, $[(x-\mathbf{p})^+] \subseteq I$. By the definition of q_κ, $q_\kappa([x]) = q_\kappa(\bigvee\{[(x-\mathbf{p})^+] : 0 < p \leq 1\}) \leq q_\kappa(I)$. So $q_\kappa(\overline{I}) = \bigvee_{x\in\overline{I}} q_\kappa([x]) \leq q_\kappa(I)$. Thus $q_\kappa(\overline{I}) = q_\kappa(I)$.

(2) Consider the onto frame map $c_M : \mathcal{L}(M) \to C\mathcal{L}(M)$. Since $c_M \circ \iota_M = c\iota_M$ is a regular cozero map, by Lemma 5.1, there exists an onto frame map $\widetilde{c_M} : (\mathcal{L}(M))_\kappa \to C\mathcal{L}(M)$ such that $\widetilde{c_M} \circ q_\kappa = c_M$. It is enough to show that $\widetilde{c_M}$ is a monomorphism. Let $a, b \in (\mathcal{L}(M))_\kappa$ be such that $\widetilde{c_M}(a) = \widetilde{c_M}(b)$. There exist $I, J \in \mathcal{L}(M)$ such that $q_\kappa(I) = a$ and $q_\kappa(J) = b$. So,

$$a = q_\kappa(I) = q_\kappa(\overline{I}) = q_\kappa(c_M(I)) = q_\kappa(\widetilde{c_M} \circ q_\kappa(I)) = q_\kappa(\widetilde{c_M}(a)) =$$

$$q_\kappa(\widetilde{c_M}(b)) = q_\kappa(\widetilde{c_M} \circ q_\kappa(J)) = q_\kappa(c_M(J)) = q_\kappa(\overline{J}) = q_\kappa(J) = b.$$

Hence $a = b$. Therefore $\widetilde{c_M}$ is an isomorphism. □

Remark 5.8. By Lemma 5.7(2), the regularization of ι_M is isomorphic to $c\iota_M$. Since $\widetilde{c_M} \circ \mathbf{R}\iota_M = \widetilde{c_M} \circ q_\kappa \circ \iota_M = c_M \circ \iota_M = c\iota_M$, $\widetilde{c_M}$ is an isomorphism frame map. So, $(id_M, \widetilde{c_M}) : \mathbf{R}\iota_M \to c\iota_M$ is an isomorphism of cozero maps.

6 Cozero of fractions and slim regularization

To give a slim regularization of a cozero map $c : M \to L$, M must be a bounded ℓ-module over a $\mathbb{Q}$-algebra ordered ring A.

Now, suppose that A is an ordered ring. Consider the ring of fractions $\breve{A} = S^{-1}A$ which is a $\mathbb{Q}$-algebra, where $S = \{n.1 : n = 1, 2, 3, \cdots\}$. For a bounded f-module M, $\breve{M} = S^{-1}M$ is a bounded f-module over $\breve{A}$ [10].

Define $\breve{c} : \breve{M} \to L$ by $\breve{c}(x/s) = c(x)$. Thus $\breve{c}$ is a well-defined cozero map by the following lemma.

Proposition 6.1. *Let M be an f-module over an ordered ring A, and $c : M \to L$ be a cozero map. Then,*

(1) *For every $n \in \mathbb{N}, c(nx) = c(x)$.*

(2) *$\breve{c} : \breve{M} \to L$ is a well-defined cozero map.*

(3) *If c is slim, then so is $\breve{c}$.*

(4) *If A is a $\mathbb{Q}$-algebra, then $\breve{c}$ is isomorphic to c.*

Proof. (1) Let $n \in \mathbb{N}$ and $x \in M$. Then,

$$c(nx) = c(|nx|) = c(n|x|) = c(|x|) \vee \cdots \vee c(|x|) = c(|x|) = c(x).$$

(2) Suppose that $x/s = y/t$. Thus, there is a $w \in S$ such that $w(tx - sy) = 0$. Hence $c(wtx) = c(wsy)$. Thus, by (1),

$$\breve{c}(x/s) = c(x) = c(wtx) = c(wsy) = c(y) = \breve{c}(y/t).$$

Therefore, $\breve{c}$ is well-defined. To show the properties of a cozero map, let $x/s, y/t \in \breve{M}$ and $a/r \in \breve{A}$. Hence

$$\breve{c}(x/s + y/t) = \breve{c}(tx + sy/st) = c(tx + sy) \leq$$

$$c(tx) \vee c(sy) = c(x) \vee c(y) = \breve{c}(x/s) \vee \breve{c}(y/t)$$

$$\breve{c}((a/r)(x/s)) = \breve{c}(ax/rs) = c(ax) \leq c(x) = \breve{c}(x/s).$$

If $x, y \geq 0$, we have

$$\breve{c}(x/s + y/t) = \breve{c}(tx + sy/st) = c(tx + sy)$$

$$= c(tx) \vee c(sy) = c(x) \vee c(y) = \breve{c}(x/s) \vee \breve{c}(y/t).$$

(3) is clear.

(4) $a \rightsquigarrow a/1$ is an isomorphism from A to $\breve{A}$. Also we have $M \simeq \breve{M}$. So $\breve{c} \simeq c$, and the proof is complete. $\square$

Remark 6.2. To obtain the regularization of c, by Remark 5.4, consider the regular cozero map $q_{\breve{c}} \circ \breve{c} : \breve{M} \to L \to L_{\breve{c}}$. So, for an arbitrary cozero map $c : M \to L$, to obtain the slim regularization of c, let $I^* = \{x/s \in \breve{M} : q_{\breve{c}} \circ \breve{c}(x/s) = \bot\}$. Consider $c^* : \breve{M}/I^* \to L_{\breve{c}}$, given by $c^*(\overline{x/s}) = q_{\breve{c}} \circ \breve{c}(x/s) = q_{\breve{c}}(c(x))$. Then, c^* is the slim regularization of c. Also, we have the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{c} & L \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle id} \\ \breve{M} & \xrightarrow{\breve{c}} & L \\ {\scriptstyle \gamma}\big\downarrow & & \big\downarrow{\scriptstyle q_{\breve{c}}} \\ \breve{M}/I^* & \xrightarrow{c^*} & L_{\breve{c}} \end{array}$$

Remark 6.3. (1) The frame $C\mathcal{L}(M)$, which is called the *Maximal Spectrum* of M, plays an important role in the construction of pointfree versions of Gelfand duality [4] and Kakutani duality [11]. In fact, these advantages of the maximal spectrum are due to the desirable properties of cozero maps $\mathbf{c}\iota : M \to C\mathcal{L}(M)$. In [17] it is shown that being slim, regularity, and algebraicity are most important advantages of the cozero map $\mathbf{c}\iota$. Because for a bounded f-module M over a $\mathbb{Q}$-algebra ordered ring and a compact frame L, every algebraic slim regular cozero map $c : M \to L$ is equal to $\mathbf{c}\iota$, up to isomorphism.

(2) By the notion of the *cozero transformation* introduced in [17], the functor $L \rightsquigarrow C(L)$ is described uniquely by the properties of being slim, regular, and algebraic. In fact, the functor $\mathbf{C} : \mathbf{KCRFrm} \to \mathbf{BfCat}$ is the unique functor (up to isomorphism) with a natural uniformly continuous slim algebraic regular cozero transformation $\mathbf{coz} : \mathbf{C} \to \mathbf{id_{KCRFrm}}$ (see Theorems 3.7 and 3.10 of [17]). This can be a categorical description of $C(L)$ as bounded f-modules over $\mathbb{Q}$-algebra ordered rings.

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