The $\lambda$-super socle of the ring of continuous functions

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Dedicated to Professor Bernhard Banaschewski on the occasion of his 90th birthday,
and for his great achievements in mathematics

Abstract. The concept of $\lambda$-super socle of $C(X)$, denoted by $S_\lambda(X)$ (that is, the set of elements of $C(X)$ such that the cardinality of their cozerosets are less than $\lambda$, where $\lambda$ is a regular cardinal number with $\lambda \leq |X|$) is introduced and studied. Using this concept we extend some of the basic results concerning $SC_F(X)$, the super socle of $C(X)$ to $S_\lambda(X)$, where $\lambda \geq \aleph_0$. In particular, we determine spaces $X$ for which $SC_F(X)$ and $S_\lambda(X)$ coincide. The one-point $\lambda$-compactification of a discrete space is algebraically characterized via the concept of $\lambda$-super socle. In fact we show that $X$ is the one-point $\lambda$-compactification of a discrete space $Y$ if and only if $S_\lambda(X)$ is a regular ideal and $S_\lambda(X) = O_x$, for some $x \in X$.

1 Introduction

The reader is referred to [7], [9], and [14] for the necessary notations, definitions, and background concerning the topological spaces $X$ and $C(X)$, the

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ring of real valued continuous functions on a space $X$. All topological spaces $X$ in this paper are Tychonoff, unless otherwise mentioned. We remind the reader that $C_F(X)$ is the socle of $C(X)$, (that is, the sum of all minimal ideals of $C(X)$ which is also the intersection of all essential ideals in $C(X)$). We should also recall that an ideal in a commutative ring is essential if it intersects every nonzero ideal of the ring nontrivially. $C_F(X)$ is introduced and topologically characterized in [19]. Recently in [13], $SC_F(X)$, the super socle of $C(X)$ has also been introduced and studied.

We know that one of the main objectives of working in the context of $C(X)$ is to characterize topological properties of a given space $X$ in terms of a suitable algebraic properties of $C(X)$. It turns out, $C_F(X)$ and $SC_F(X)$ play an appropriate role, with respect to this objective, in the literature, see [1], [2], [10], [13], [17], and [18]. The importance of the role of $C_F(X)$ and $SC_F(X)$ in the context of $C(X)$, motivated us to define and study a general concept of the socle of $C(X)$, called $\lambda$-super socle, which includes the latter two socles.

An outline of this article is as follows: In Section 2, the concept of the $\lambda$-super socle and some preliminary results concerning this ideal, which are frequently used in the subsequent sections, are given. In particular, we characterize topological spaces $X$ such that $\lambda$-super socle and $C_F(X)$ or $SC_F(X)$ coincide. We also present a characterization of the one-point $\lambda$-compactification of discrete spaces in terms of the $\lambda$-super socle. In the final section, the $\lambda$-pseudo minimal ideals and $\lambda$-disjoint spaces are introduced and it is shown that for these spaces, $S_\lambda(X)$ can be written in a form of direct sum (called $\lambda$-strong direct sum) of certain subideals.

2 The $\lambda$-super socle of $C(X)$

Let us, without further ado, begin by formally defining the $\lambda$-super socle of $C(X)$, the extension of super socle of $C(X)$ (that is, the set $SC_F(X) = \{f \in C(X) : X \setminus Z(f) \text{ is countable}\}$) which is introduced in [13].

**Definition 2.1.** The set $S_\lambda(X) = \{f \in C(X) : |Coz(f)| < \lambda\}$, where $|Coz(f)| = |X \setminus Z(f)|$ and $\lambda$ is a regular cardinal number with $\lambda \leq |X|$, is called the $\lambda$-super socle of $C(X)$.

By convention, we put $S_\mu(X) = C(X)$, where $\mu$ is a regular cardinal number greater than $|X|$. One can easily show that $S_\lambda(X)$ is a z-ideal
in \(C(X)\) and \(SC_F(X) \subseteq S_\lambda(X)\), where \(\lambda \geq \aleph_1\). Manifestly \(SC_F(X) = S_{\aleph_1}(X)\) and \(S_{\aleph_1}(X) = C(X)\) if and only if \(X\) is a countable space, see [13]. Clearly \(C_F(X) \subseteq S_\lambda(X)\), where \(\lambda \geq \aleph_0\). In view of [18, Proposition 3.3], or by using some other well-known algebraic methods, one can easily see that \(C_F(X) = C(X)\) if and only if \(X\) is a finite space. It is also easy to observe that if \(X\) is an infinite discrete space, then \(C(X) \cong \prod_{x \in X} R_x\), where \(R_x = \mathbb{R}\). Moreover, \(C_F(X) \cong \bigoplus_{x \in X} R_x\), by [18, Proposition 3.3]. Let us recall that if \(a = (a_i)\) is an element of \(\prod_{i \in I} R_i\), where each \(R_i\) is an arbitrary ring, then the support of \(a\), which is denoted by \(\text{supp}(a)\), is defined by \(\text{supp}(a) = \{i \in I : a_i \neq 0\}\). Consequently, \(S_\lambda(X)\) is in one to one correspondence with the set of the elements of \(\lambda\)-support (that is, \(|\text{supp}(a)| < \lambda\)), in \(\prod_{x \in X} R_x\), where \(R_x = \mathbb{R}\). It is trivial to see that a point in a space \(X\) is isolated if and only if it has a finite neighborhood. If \(|X| = \lambda\) and \(\aleph_0 = \aleph_0 < \aleph_1 = \lambda_1 < \ldots < \lambda^+\) is a chain of regular cardinal numbers then we have

\[C_F(X) = S_{\aleph_0}(X) \subseteq SC_F(X) = S_{\lambda_1}(X) \subseteq \ldots \subseteq S_{\lambda^+}(X) = C(X)\]

It is also manifest that if \(|X| = \lambda\), where \(\lambda\) is regular then \(S_\lambda(X)\) is the largest proper ideal among all \(\mu\)-supersocles (note, we may have \(S_\lambda(X) = 0\)).

Motivated by this, the next two definitions are natural and are also needed.

**Definition 2.2.** An element \(x \in X\) is called a \(\lambda\)-isolated point if \(x\) has a neighborhood with cardinality less than \(\lambda\). The set of \(\lambda\)-isolated points of \(X\) is denoted by \(I_\lambda(X)\).

**Definition 2.3.** A space \(X\) is called \(\lambda\)-discrete if \(I_\lambda(X) = X\).

We note that \(W(\lambda)\), the space of all ordinals less than \(\lambda\), where \(\lambda\) is a cardinal number, is a \(\lambda\)-discrete space, see [14, 5.11]. Clearly, a point is isolated if and only if it is \(\aleph_0\)-isolated, and the set of all isolated points of \(X\) is denoted by \(I(X)\). We should also remind the reader that \(I_{\aleph_1}(X)\) is denoted by \(I_c(X)\). We should also recall here that a subspace of an \(\aleph_1\)-discrete space is countable if and only if it is Lindelöf, see [10]. Similarly, a subspace of a \(\lambda^+\)-discrete space has the cardinality \(\lambda\) if and only if it is \(\lambda\)-compact.

Evidently, every space with the cardinality \(\lambda\) is a \(\lambda^+\)-discrete space, and any finite direct product of \(\lambda\)-discrete spaces is \(\lambda\)-discrete. It also goes
without saying that a subspace of a \( \lambda \)-discrete space is \( \lambda \)-discrete. Clearly, if \( X = \prod_{s \in S} X_s \) is \( \lambda \)-discrete, then each \( X_i \) is \( \lambda \)-discrete too, but the converse is not necessarily true. It is also manifest that the free union \( X = \bigoplus_{s \in S} X_s \) is \( \lambda \)-discrete if and only if each \( X_s \) is \( \lambda \)-discrete for each \( s \in S \).

Let us recall the concept of \( \lambda \)-compactness in [17].

**Definition 2.4.** A topological space \( X \) is called \( \lambda \)-compact if each open cover of \( X \) can be reduced to an open cover whose cardinality is less than \( \lambda \), where \( \lambda \) is the least infinite cardinal number with this property.

The following result is evident.

**Proposition 2.5.** In a \( \lambda \)-discrete space, every \( \lambda \)-compact subspace has cardinality less than \( \lambda \).

The following lemma whose proof can be given using the proof of [13, Proposition 2.4], word for word, is needed.

**Lemma 2.6.** For any space \( X \),

\[ I_\lambda (X) = \bigcup \{ \text{coz}(f) : f \in S_\lambda (X) \}. \]

We recall that the ideal \( I \) of \( C(X) \) is free if \( \bigcap_{f \in I} Z(f) = \emptyset \), that is, \( \bigcup \{ \text{coz}(f) : f \in I \} = X \).

The following result is now immediate.

**Corollary 2.7.** For any space \( X \), the following statements hold:

1. The ideal \( S_\lambda (X) \) is not a zero ideal if and only if \( X \) has a \( \lambda \)-isolated point.
2. The space \( X \) is a \( \lambda \)-discrete space if and only if \( S_\lambda (X) \) is free.
3. For each \( x \in X \), \( M_x = \{ f \in C(X) : f(x) = 0 \} \) is a maximal ideal.

**Corollary 2.8.** For any space \( X \) we have the following:

1. An element \( x \) is a \( \lambda \)-isolated point if and only if \( M_x + S_\lambda (X) = C(X) \).
2. \( X \) is a \( \lambda \)-discrete space if and only if for all \( x \in X \), \( M_x + S_\lambda (X) = C(X) \).
3. The ideal \( S_\lambda (X) \) is a free ideal in \( C(X) \) if and only if for all \( x \in X \), \( M_x + S_\lambda (X) = C(X) \).
4. An element \( x \) is non-\( \lambda \)-isolated point if and only if \( S_\lambda (X) \subseteq M_x \).
5. Let \( X \) be a topological space with \( |X| \geq \lambda \) and \( |I_\lambda (X)| < \lambda \). Then \( S_\lambda (X) = \bigcap_{x \in X \setminus I_\lambda (X)} M_x \).
Proof. We only give the proofs of parts (1) and (5).

(1) Let \( x \in X \) be a \( \lambda \)-isolated point. Then by Lemma 2.6, there exists \( f \in S_\lambda(X) \) such that \( f(x) = 1 \). So \( (1-f) \in M_x \), hence \( S_\lambda(X) + M_x = C(X) \). Now let \( M_x + S_\lambda(X) = C(X) \). Then there exists \( h \in S_\lambda(X) \) such that \( (1-h) \in M_x \). This implies that \( x \in X \setminus Z(h) \), where \( |X \setminus Z(h)| < \lambda \). Consequently, \( x \) is a \( \lambda \)-isolated point.

(5) By part (4) and our assumption, \( S_\lambda(X) \subseteq \bigcap_{x \notin I_\lambda(X)} M_x \). Now we may assume that \( 0 \neq f \in \bigcap_{x \notin I_\lambda(X)} M_x \). Hence \( x \in X \setminus I_\lambda(X) \subseteq Z(f) \), and since \( |I_\lambda(X)| < \lambda \), we infer that \( f \in S_\lambda(X) \) and we are done. \( \square \)

The following is an extension of [13, Theorem 2.7].

**Theorem 2.9.** \( I_\lambda(X) \) is finite if and only if \( S_\lambda(X) = C_F(X) \), where \( \lambda \geq \aleph_1 \). In particular, in this case, \( SC_F(X) = C_F(X) \).

**Proof.** \((\Rightarrow)\) If \( I_\lambda(X) \) is finite then \( x \) is isolated for each \( x \in I_\lambda(X) \). So \( I_\lambda(X) = I(X) \), and consequently \( S_\lambda(X) = C_F(X) \), see also [19].

\((\Leftarrow)\) Suppose \( S_\lambda(X) = C_F(X) \) and \( I_\lambda(X) \) is an infinite set, and seek a contradiction. Let \( C = \{x_1, x_2, \ldots\} \subseteq I_\lambda(X) \) be a countable subset. Hence for each \( x_n \in C \), there exists an open set \( G_n \), with the cardinality less than \( \lambda \). By completely regularity of \( X \), for each \( n \geq 1 \) there exists \( f_n \in C(X) \), such that \( f_n(x_n) = 1 \) and \( f_n(X \setminus G_n) = (0) \). Now put \( f = \sum_{n=1}^{\infty} \frac{f_n^2}{f_n^2 + 2^{-n}} \), and note that for each \( n \geq 1 \), \( f(x_n) \neq 0 \), and consequently \( f \notin C_F(X) \), see [19]. But we claim that \( f \in S_\lambda(X) \). To see this, it is enough to show that \( |X \setminus Z(f)| < \lambda \). Hence it suffices to show that \( X \setminus Z(f) \subseteq \bigcup_{n=1}^{\infty} G_n \).

Let \( x \in X \setminus Z(f) \) and \( x \notin \bigcup_{n=1}^{\infty} G_n \). Thus \( x \in \bigcap_{n=1}^{\infty}(X \setminus G_n) \) and \( f(x) = 0 \), which is the desired contradiction. \( \square \)

The following proposition is evident (note that, if \( \lambda > \aleph_0 \) then \( I_c(X) = I_{\aleph_1}(X) \subseteq I_\lambda(X) \)).

**Proposition 2.10.** Let \( \lambda > \aleph_0 \) and \( |I_\lambda(X)| \leq \aleph_0 \). Then \( S_\lambda(X) = SC_F(X) \).

We recite the following definition from [15].

**Definition 2.11.** Let \((Y, \tau)\) be an uncountable discrete space with cardinality greater than or equal to \( \lambda \) such that \( \lambda \) is regular cardinal number. Similar to the one-point compactification construction of \( Y \), put \( X = Y \cup \{x\} \), where \( x \) is not a point in \( Y \). Let \( \tau^* = \tau \cup \{G \subseteq X : x \in G, |X \setminus G| < \lambda\} \). It is
clear that \((X, \tau^*)\) is a Hausdorff and \(\lambda\)-compact space. This space is called the one-point \(\lambda\)-compactification of the discrete space \(Y\).

It is easy to check that the space \(X\) above, is completely regular. To this end, we just recall that a Hausdorff space whose set of nonisolated points is finite, is normal, see [9]. Moreover, since \(\lambda\) is a regular cardinal number, one can show that for any cardinal number \(\gamma < \lambda\), \(\bigcap_{i \in I} G_i\) is open, where every \(G_i\) is open in \(X\) and \(|I| \leq \gamma\), that is, \(X\) is a \(P\lambda\)-space (note, \(X\) is \(P\lambda\)-space if every intersection with cardinality less than \(\lambda\) of open sets (that is, \(G\lambda\)-set) is open). \((X, \tau^*)\) with this structure is a non \(\lambda\)-discrete space and \(I\lambda(X) = Y\).

The next remark gives an example of a non \(\lambda\)-discrete space \(X\) for which \(S\lambda(X) \neq SC\lambda(X)\), where \(\lambda\) is an infinite cardinal number.

**Remark 2.12.** Let \(X = Y \cup \{x\}\) be the one-point \(\lambda\)-compactification of a discrete space \(Y\), with \(|Y| \geq \lambda \geq \aleph_1\). Let us define the map \(f\) as follows,

\[
f(x) = \begin{cases} 
0, & x \in G \\
1, & x \in X \setminus G 
\end{cases}
\]

where \(G\) is an open set containing \(x\) with \(\aleph_0 < |X \setminus G| < \lambda\). Hence \(f \in S\lambda(X) \setminus SC\lambda(X)\). This shows that \(X\) is a non \(\lambda\)-discrete space with \(S\lambda(X) \neq SC\lambda(X)\).

Let us recall that every \(\lambda\)-discrete space \(X\) is a locally \(\lambda\)-compact (a Hausdorff space \(X\) is called locally \(\lambda\)-compact if every \(x \in X\) has a neighborhood which is \(\lambda\)-compact). An ideal \(I\) in a commutative ring \(R\) is called regular if for each \(a \in I\) there exists \(b \in I\) such that \(a = a^2b\). It is well-known and easy to prove that a ring \(R\) is regular if and only if there is a regular ideal \(I\) in \(R\) such that \(\frac{R}{I}\) is regular, too, see [15, Lemma 1.3].

We conclude this section with the following theorem, which is our main result.

Let us recall that \(O_x = \{f \in C(X) : Z(f) \text{ is a neighborhood of } x\}\), which is a fixed ideal in \(C(X)\).

The following lemma, which is needed for the proof of our main theorem, generalizes the well-known fact that every countable subset of a \(P\lambda\)-space is closed discrete, see [14].
Lemma 2.13. If $X$ is a $P_{\lambda}$-space, then every subset of cardinality less than 
$\lambda$ is a closed discrete subspace.

Proof. Let $Y \subseteq X$ with $|Y| < \lambda$ and $x \in Y$. Clearly, for each $y \in Y \setminus \{x\}$, 
there exists $f_y \in C(X)$ such that $f_y(x) = 0$, $f_y(y) = 1$. Therefore $x \in \bigcap_{y \in Y \setminus \{x\}} Z(f_y) = G$. Since $|Y \setminus \{x\}| < \lambda$ and $X$ is a $P_{\lambda}$-space, we infer that 
$G$ is an open subset of $X$. It goes without saying that $\{x\} = G \cap Y$ which means that $\{x\}$ is open in $Y$, and consequently $Y$ is discrete. Moreover, $Y$ is closed, by \cite[Remark 1.12]{17}.

Theorem 2.14. The following statements are equivalent for a space $X$ with $|X| \geq \lambda$, where $\lambda \geq \aleph_1$ is a regular cardinal.

(1) $X$ is the one-point $\lambda$-compactification of a discrete space.

(2) $X$ is a $P_{\lambda}$-space and $S_\lambda(X) = O_x$, for some $x \in X$.

Proof. (1) $\Rightarrow$ (2) Let $X = Y \cup \{x\}$ be the one-point $\lambda$-compactification of 
$Y$, where $Y$ is discrete. By \cite[Example 1.8]{17}, $X$ is a $P_{\lambda}$-space (note, $X$ is 
a $P_{\lambda}$-space if for any $\gamma < \lambda$, $\bigcap_{i \in I} G_i$ is open, where each $G_i$ is open and 
$|I| \leq \gamma$), (we should emphasize that $X$ is a $P_{\lambda}$-space too). Thus it remains 
to be shown that $S_\lambda(X) = O_x$. First, we prove that $S_\lambda(X) \subseteq O_x$. To see 
this, let $f \in S_\lambda(X)$ and note that $|X \setminus Z(f)| < \lambda$. Hence $x \notin X \setminus Z(f)$, 
for $|Z(f)| \geq \lambda$. Since $X$ is a $P_{\lambda}$-space, we infer that $Z(f)$ is open, therefore 
f $\in O_x$. Then $Z(f)$ is a neighborhood of $x$, and therefore 
by definition $|X \setminus Z(f)| < \lambda$, that is, $f \in S_\lambda(X)$, and we are done.

(2) $\Rightarrow$ (1) Let $O_x = S_\lambda(X)$, for some $x \in X$, where $X$ is a $P_{\lambda}$-space. 
Put $Y = X \setminus \{x\}$. First, we show that $x$ is a non $\lambda$-isolated point, a fortiori 
nonisolated point. To see this, let $G$ be an open set containing $x$ whose 
cardinality is less than $\lambda$ and get a contradiction. By complete regularity 
of $X$, there exist $f, g \in C(X)$ such that $Z(f) \cap Z(g) = \emptyset$, $x \in \text{int}Z(f)$, and 
$X \setminus G \subseteq \text{int}Z(g)$, see \cite[Theorem 1.15]{14}. Since $f \in O_x = S_\lambda(X)$, we infer 
that $|X \setminus Z(f)| < \lambda$. We notice that $X \setminus G \subseteq Z(g)$ and $Z(f) \subseteq X \setminus Z(g) \subseteq G$. 
Consequently, $|Z(f)| < \lambda$ which shows that $|X| < \lambda$, that is absurd. Now 
we claim that $x$ is the only non $\lambda$-isolated point of $X$. If not, let $y \neq x$ 
be another non $\lambda$-isolated point in $X$ and seek a contradiction. Again by 
complete regularity of $X$, there exist $f, g \in C(X)$ such that $Z(f) \cap Z(g) = \emptyset$, 
x $\in \text{int}Z(f)$, and $y \in \text{int}Z(g)$. This implies that $f \in O_x = S_\lambda(X)$ and 
$|X \setminus Z(f)| < \lambda$. But $Z(g) \subseteq X \setminus Z(f)$ implies that $|Z(g)| < \lambda$. So $y$ is a $\lambda$- 
isolated point, which is a contradiction. Now we prove that the cardinality
of the complement of each open set containing \( x \) is less than \( \lambda \). To see this, let \( G \) be an open set containing \( x \). Then there exists \( f \in C(X) \) such that \( x \in Z(f) \subseteq G \) (note, \( Z(f) \) is open, since \( X \) is a \( P_\lambda \)-space). This implies that \( f \in O_x = S_\lambda(X) \), hence \( |X \setminus G| \leq |X \setminus Z(f)| < \lambda \) and we are done. It remains to show that each \( y \in Y \) is an isolated point, and every subset \( G \) containing \( x \) with \( |X \setminus G| < \lambda \) is an open neighborhood of \( x \). First, we show that each \( y \in Y \) is an isolated point of \( X \). We have already shown that \( y \) is a \( \lambda \)-isolated point in \( X \) which means that there is an open set \( H \) containing \( y \) such that \( |H| < \lambda \). In view of the previous lemma, \( H \) is a discrete closed subspace of \( X \). Since \( H \) is open, we infer that all of its points are isolated, hence \( y \) is isolated. We notice that we have already shown that the above set \( G \), where \( |X \setminus G| < \lambda \) is a neighborhood of \( x \).

It is interesting to note that in the proof of the part \((2) \Rightarrow (1)\) of the previous theorem, instead of the assumption \( O_x = S_\lambda(X) \), we just used the fact that \( O_x \subseteq S_\lambda(X) \). As an immediate consequence of Theorem 2.14, we obtain that if in a \( P_\lambda \)-space \( X \) with \( |X| \geq \lambda \), \( O_x \subseteq S_\lambda(X) \), for some \( x \in X \), then \( O_x = S_\lambda(X) \) and \( X \) is the one-point \( \lambda \)-compactification of a discrete space \( Y \) with \( |Y| \geq \lambda \). It is known that \( C_F(X) \) is never a prime ideal in \( C(X) \), see [10, Proposition 1.2]. In contrast to this fact, it has already been observed that \( SC_F(X) \) can be a prime ideal (even a maximal ideal). Let us record this fact which is an advantage of \( S_\lambda(X) \) over \( C_F(X) \), where \( \lambda > \aleph_0 \), in the context of \( C(X) \). The following corollary is proved in [13, 2.19].

**Corollary 2.15.** Let \( X \) be either countable or one-point \( \aleph_1 \)-compactification of some uncountable discrete space. Then \( SC_F(X) = S_{\aleph_1}(X) \) is a prime ideal in \( C(X) \).

We immediately have the following proposition, see also [17, Example 1.8].

**Proposition 2.16.** Let \( X \) be the one-point \( \lambda \)-compactification of a discrete space \( Y \) with \( |Y| > \lambda \). Then \( S_\lambda(X) \) is a prime ideal (in fact a maximal ideal) in \( C(X) \).

3 \( \lambda \)-Pseudo minimal ideals and \( \lambda \)-disjoint spaces

In this section we are trying to extend the definitions and the results of [13, Section 4] concerning the super socle of \( C(X) \) to \( S_\lambda(X) \). We recall that
\( C_F(X) \) is a direct sum of minimal ideals in \( C(X) \), which are evidently generated by idempotents. Note that, if \( I \) is a minimal ideal in \( C(X) \) then \( I = eC(X) \), where \( e \in C(X) \) is an idempotent such that there is \( x \in X \) with \( e(x) = 1 \) and \( e(X \setminus \{x\}) = 0 \) (clearly, \( x \) is an isolated point in \( X \), see [19, Proposition 3.1]). Similarly to \( C_F(X) \), in [13, lemma 4.2], it is shown that the super socle of \( C(X) \) is a kind of direct sum of ideals in \( C(X) \), which is not necessarily a direct sum. Motivated by this, we show that similar results hold for \( S_\lambda(X) \), too.

The following definition is the counterpart of [13, Definition 4.1].

**Definition 3.1.** Let \( G \) be an open neighborhood of \( x \in X \) with \( |G| < \lambda \). Then the ideal \( (f^x_G) \), where \( f^x_G \in C(X) \) such that \( f^x_G(x) = 1 \) and \( f^x_G(X \setminus G) = (0) \) (note, by complete regularity of \( X \), \( f^x_G \) exists and \( f^x_G \in S_\lambda(X) \), but it is not necessarily unique) is called a \( \lambda \)-pseudo minimal ideal at \( x \).

Let \( \mathcal{L}(X) \) be the set of all open subsets of \( X \) with cardinality less than \( \lambda \). Take \( G \in \mathcal{L}(X) \) and \( x \in G \). Then we say that an element \( f \in C(X) \) is of the form \( f^x_G \), if \( f(x) = 1 \) and \( f(X \setminus G) = 0 \). Now put

\[
F^x_G = \{ f \in C(X) : f \text{ is of the form } f^x_G \}
\]

and \( F^x_{\mathcal{L}(X)} = \bigcup G \in \mathcal{L}(X) \) \( F^x_G \). Then \( S_x = \sum_{f \in F^x_{\mathcal{L}(X)}} (f) \) is called the \( \lambda \)-pseudo socle at \( x \). Finally, if we put

\[
F^{I_\lambda(X)} \mathcal{L}(X) = \{ f \in C(X) : f \text{ is of the form } f^x_G, \text{ where } (x,G) \in I_\lambda(X) \times \mathcal{L}(X) \text{ and } x \in G \}
\]

and \( S = \sum_{f \in F^{I_\lambda(X)} \mathcal{L}(X)} (f) \), then \( \sum_{x \in I_\lambda(X)} S_x = S \subseteq S_\lambda(X) \) is called the \( \lambda \)-pseudo socle of \( C(X) \).

We should emphasize that \( (g) \), where \( g \in C(X) \), is a \( \lambda \)-pseudo minimal ideal at an element \( x \in I_\lambda(X) \) if and only if \( g \in F^{I_\lambda(X)} \mathcal{L}(X) \). We should also recall that if \( x \in X \) is an isolated point then the pseudo minimal ideal \( (f^x_{\{x\}}) \) at \( x \), is in fact a minimal ideal in \( C(X) \), by the comment preceding the previous definition, see also [19, Proposition 3.1]. It is clear that \( (f^x_{\{x\}}) \) is contained in every \( \lambda \)-pseudo minimal ideal, \( (f^x_G) \) say, at \( x \) (note, \( f^x_{\{x\}} f^x_G \neq 0 \) implies
that \((f^x_G)_x = (f^x_G) \subseteq (f^x_G))\). Consequently,
\[
(f^x_G) = \bigcap \{(g) : (g) \text{ is a } \lambda\text{-pseudo minimal ideal at } x\}
\]
(that is, every minimal ideal in \(C(X)\), which is clearly a \(\lambda\)-pseudo minimal ideal at an isolated point \(x \in X\), is the intersection of all \(\lambda\)-pseudo minimal ideals at \(x\)), see also [13].

The following lemma is now evident (note, let \(0 \neq f \in S_\lambda(X)\), \(f(x) = r\), for some \(x \in X \setminus Z(f) = G\), then \(f \in (f^x_G) \subseteq S_\lambda(X) \subseteq S\), where \(f^x_G = r^{-1} f\)).

**Lemma 3.2.** \(S_\lambda(X) = \sum_{x \in I_\lambda(X)} S_x = S\) (that is, the \(\lambda\)-super socle and the \(\lambda\)-pseudo socle of \(C(X)\) coincide).

It is manifest that the above sum is not necessarily a direct sum of ideals \(S_x\). Next, we are looking for spaces \(X\) in order to get some kind of direct sum for \(S_\lambda(X)\). Let us begin with an example as a prototype.

The following example imitates [13, Example 4.3].

**Example 3.3.** Let \(X\) be a discrete space such that \(|X| \geq \lambda\). In view of [19, Proposition 3.3] and its proof, we have \(C_F(X) = \sum_{x \in X} \bigoplus (f^x_G)\). As for the \(\lambda\)-super socle, first we may put \(X = \bigcup_{i \in I} X_i\), where each \(X_i\) is an infinite subset of \(X\) with cardinality less than \(\lambda\), and for \(i \neq j\), \(X_i \cap X_j = \emptyset\). Now for each \(i \in I\) we define the ideal \(S_i = \{f \in C(X) : X \setminus Z(f) \subseteq X_i\}\). Then one easily shows that \(S_\lambda(X) = \sum_{i \in I} \bigcup S_i\), where an element \(f\) of the latter sum is of the form \(f = \sum_{i \in I} f_i\) with \(f_i \in S_i\) such that \(|\{i \in I : f_i \neq 0\}| < \lambda\), (note that the infinite sum \(f = \sum_{i \in I} f_i\) is well-defined, for, if \(x \in X\) then \(f(x) = f_i(x)\) for a unique \(i \in I\)). It is manifest that \(\sum_{i \in I} \bigoplus S_i\) exists and it is a subideal of \(S_\lambda(X)\).

If \(\sum_{i \in I} \bigoplus S_i\) and \(\sum_{i \in I} \bigcup S_i\) exist for a collection of ideals \(S_i\) in \(C(X)\), then similar to [13], we call the latter sum “\(\lambda\)-strong direct sum” of these ideals.

Motivated by the previous example, we present the next theorem, which was promised earlier. First we need the following definition.

**Definition 3.4.** A space \(X\) is called \(\lambda\)-disjoint, if its \(\lambda\)-isolated points (that is, \(I_\lambda(X)\)) can be disjointly separated, that is, its \(\lambda\)-isolated points can
be written as a union of disjoint collection of clopen subsets of $X$ with cardinality less than $\lambda$.

**Example 3.5.** Discrete spaces with cardinality greater than or equal to $\lambda$, a topological space $X$ without $\lambda$-isolated points (for example, $X = \mathbb{R}$ with the usual topology), the one-point $\lambda$-compactification of a discrete space $D$, where $|D| \geq \lambda$, the sum (that is, free union) of any collection of $\lambda$-disjoint spaces, $c$-disjoint spaces, where $c$ is the cardinality of $\mathbb{R}$, $c \leq \lambda$, and finally $X = Y \oplus Z$, where $Y$ has no $\lambda$-isolated points and $Z$ is a $\lambda$-disjoint space, are some examples of $\lambda$-disjoint spaces.

We conclude this section by proving that for $\lambda$-disjoint spaces, the $\lambda$-super socle of $C(X)$ is almost decomposable (that is, it is a $\lambda$-strong direct sum or a direct sum of some of its subideals).

**Theorem 3.6.** Let $\lambda$ be a regular cardinal number and $X$ be a $\lambda$-disjoint space with $X = Y \cup Z$, $Y \cap Z = \emptyset$ such that $I_\lambda(X) = Z = \bigcup_{i \in I} G_i$, where each $G_i$ is a clopen set with cardinality less than $\lambda$, and $G_i \cap G_j = \emptyset$ for all $i \neq j$. Then $S_\lambda(X) = \sum_{i \in I} \bigoplus S_i$, where $S_i = \{ f \in C(X) : X \setminus Z(f) \subseteq G_i \}$. Moreover, $\sum_{i \in I} \bigoplus S_i \subseteq S_\lambda(X)$, and if $I$ is finite then $S_\lambda(X) = \sum_{i \in I} \bigoplus S_i$.

**Proof.** Let $f \in \sum_{i \in I} \bigoplus S_i$. Then $f = \sum_{i \in I} f_i$ with $f_i \in S_i$. This implies that $X \setminus Z(f) \subseteq \bigcup_{i \in J} X \setminus Z(f_i)$, where $J \subseteq I$ with $|J| < \lambda$ such that $f_i \neq 0$ for all $i \in J$. But for each $i$, $X \setminus Z(f_i) \subseteq G_i$ implies $|X \setminus Z(f_i)| < \lambda$, and consequently $|X \setminus Z(f)| < \lambda$. Hence $f \in S_\lambda(X)$. Conversely, let $f \in S_\lambda(X)$. Now for each $i \in I$, we define $f_i \in C(X)$ by $f_i(x) = f(x)$ for each $x \in G_i$ and $X \setminus G_i \subseteq Z(f_i)$. So $X \setminus Z(f_i) \subseteq G_i$, and therefore $f_i \in S_i$, for all $i \in I$. Since $|X \setminus Z(f_i)| < \lambda$, we infer that $X \setminus Z(f) \subseteq Z$ which implies that $f(G_i) \neq 0$, where $|\{ i \in I : f(G_i) \neq 0 \}| < \lambda$. Clearly, $f(Y) = 0$. Thus, whenever $f(x) \neq 0$, there is a unique $i \in I$ with $x \in G_i$ such that $f_i(x) = f(x)$, which immediately shows that $f = \sum_{i \in I} f_i$, where $f_i \in S_i$, and we are done. The last part is evident.

**Definition 3.7.** Let $\{ A_i : i \in I \}$ be a collection of ideals in $C(X)$. If for each $f_i \in A_i$, $\sum_{i \in I} f_i \in \mathbb{R}^X$, where $(\sum_{i \in I} f_i)(x) = \sum_{i \in I} f_i(x)$ is well-defined for all
\( x \in X, \) then by the external sum of these ideals, we mean
\[
\sum_{i \in I} A_i = \{ f \in \mathbb{R}^X : f = \sum_{i \in I} f_i, f_i \in A_i \}.
\]

Clearly, \( \sum_{i \in I} A_i \) may not be an ideal in \( C(X) \) (note, it is indeed an ideal in \( C(X) \) if \( I \) is finite), but it is naturally a \( C(X) \)-module.

**Remark 3.8.** Let \( X = \bigoplus_{i \in I} X_i \) be the sum (free union) of spaces \( X_i \), where \( |X_i| \geq \lambda \), for each \( i \in I \) and \( I(X) \) be the set of isolated points of \( X \). For each \( i \in I \), define \( e_i(X) = 1 \) and \( e_i(X_j) = 0 \) for all \( j \neq i \). Then we may assume that \( C(X_i) = e_i C(X) \). Clearly, \( \sum_{i \in I} \sum_{X_i} \subseteq C(X) = \sum_{i \in I} C(X_i) \) (note, \( 1 = \sum_{i \in I} e_i \)). In view of [19, Propositions 3.1, 3.3], one can easily see that
\[
C_F(X) = \sum_{x \in I(X)} \left( \bigoplus_{\{x\}} f^*_x \right) = \sum_{i \in I} C_F(X_i).
\]
Moreover, \( S_\lambda(X) = \sum_{i \in I} \bigcup_{\lambda} S_\lambda(X_i) \).

Let us prove the latter equality. Let \( f \in S_\lambda(X) \). Then \( |X \setminus Z(f)| < \lambda \), hence \( |J| < \lambda \), where \( J = \{ i \in I : X_i \cap (X \setminus Z(f)) \neq \emptyset \} \). For each \( i \in J \) put \( G_i = X_i \cap (X \setminus Z(f)) \). Now define \( f_i = e_i f \), for each \( i \in I \) and note that \( f(G_i) \neq 0 \), whenever \( i \in J \). Clearly, \( f_i \in S_\lambda(X_i) \) for each \( i \in I \), hence \( f = \sum_{i \in I} f_i \in \bigoplus_{i \in I} S_\lambda(X_i) \). The converse is evident.

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