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The λ-super socle of the ring of continuous functions

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Dedicated to Professor Bernhard Banaschewski on the occasion of his 90th birthday, and for his great achievements in mathematics

Abstract. The concept of λ-super socle of $C(X)$, denoted by $S_\lambda(X)$ (that is, the set of elements of $C(X)$ such that the cardinality of their cozerosets are less than λ, where λ is a regular cardinal number with $\lambda \leq |X|$) is introduced and studied. Using this concept we extend some of the basic results concerning $SC_F(X)$, the super socle of $C(X)$ to $S_\lambda(X)$, where $\lambda \geq \aleph_0$. In particular, we determine spaces X for which $SC_F(X)$ and $S_\lambda(X)$ coincide. The one-point λ-compactification of a discrete space is algebraically characterized via the concept of λ-super socle. In fact we show that X is the one-point λ-compactification of a discrete space Y if and only if $S_\lambda(X)$ is a regular ideal and $S_\lambda(X) = O_x$, for some $x \in X$.

1 Introduction

The reader is referred to [7], [9], and [14] for the necessary notations, definitions, and background concerning the topological spaces X and $C(X)$, the

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ring of real valued continuous functions on a space X. All topological spaces X in this paper are Tychonoff, unless otherwise mentioned. We remind the reader that $C_F(X)$ is the socle of $C(X)$, (that is, the sum of all minimal ideals of $C(X)$ which is also the intersection of all essential ideals in $C(X)$). We should also recall that an ideal in a commutative ring is essential if it intersects every nonzero ideal of the ring nontrivially. $C_F(X)$ is introduced and topologically characterized in [19]. Recently in [13], $SC_F(X)$, the super socle of $C(X)$ has also been introduced and studied.

We know that one of the main objectives of working in the context of $C(X)$ is to characterize topological properties of a given space X in terms of a suitable algebraic properties of $C(X)$. It turns out, $C_F(X)$ and $SC_F(X)$ play an appropriate role, with respect to this objective, in the literature, see [1], [2], [10], [13], [17], and [18]. The importance of the role of $C_F(X)$ and $SC_F(X)$ in the context of $C(X)$, motivated us to define and study a general concept of the socle of $C(X)$, called λ-super socle, which includes the latter two socles.

An outline of this article is as follows: In Section 2, the concept of the λ-super socle and some preliminary results concerning this ideal, which are frequently used in the subsequent sections, are given. In particular, we characterize topological spaces X such that λ-super socle and $C_F(X)$ or $SC_F(X)$ coincide. We also present a characterization of the one-point λ-compactification of discrete spaces in terms of the λ-super socle. In the final section, the λ-pseudo minimal ideals and λ-disjoint spaces are introduced and it is shown that for these spaces, $S_\lambda(X)$ can be written in a form of direct sum (called λ-strong direct sum) of certain subideals.

2 The λ-super socle of $C(X)$

Let us, without further ado, begin by formally defining the λ-super socle of $C(X)$, the extension of super socle of $C(X)$ (that is, the set $SC_F(X) = \{f \in C(X) : X \setminus Z(f) \text{ is countable}\}$) which is introduced in [13].

Definition 2.1. The set $S_\lambda(X) = \{f \in C(X) : |Coz(f)| < \lambda\}$, where $|Coz(f)| = |X \setminus Z(f)|$ and λ is a regular cardinal number with $\lambda \leq |X|$, is called the λ-super socle of $C(X)$.

By convention, we put $S_\mu(X) = C(X)$, where μ is a regular cardinal number greater than $|X|$. One can easily show that $S_\lambda(X)$ is a z-ideal in $C(X)$ and $SC_F(X) \subseteq S_\lambda(X)$, where $\lambda \geq \aleph_1$. Manifestly $SC_F(X) = S_{\aleph_1}(X)$ and $S_{\aleph_1}(X) = C(X)$ if and only if X is a countable space, see [13]. Clearly $C_F(X) \subseteq S_\lambda(X)$, where $\lambda \geq \aleph_0$. In view of [18, Proposition 3.3], or by using some other well-known algebraic methods, one can easily see that $C_F(X)= C(X)$ if and only if X is a finite space. It is also easy to observe that if X is an infinite discrete space, then $C(X) \cong \prod_{x\in X} R_x$, where $R_x = \mathbb{R}$. Moreover, $C_F(X) \cong \sum_{x\in X} \bigoplus R_x$, by [18, Proposition 3.3]. Let us recall that if $a =< a_i >$ is an element of $\prod_{i\in I} R_i$, where each R_i is an arbitrary ring, then the support of a, which is denoted by $supp(a)$, is defined by $supp(a) = \{i \in I : a_i \neq 0\}$. Consequently, $S_\lambda(X)$ is in one to one correspondence with the set of the elements of λ-support (that is, $|supp(a)| < \lambda$), in $\prod_{x\in X} R_x$, where $R_x = \mathbb{R}$. It is trivial to see that a point in a space X is isolated if and only if it has a finite neighborhood. If $|X| = \lambda$ and $\aleph_0 = \lambda_0 < \aleph_1 = \lambda_1 < \ldots < \lambda^+$ is a chain of regular cardinal numbers then we have

$$C_F(X) = S_{\lambda_0}(X) \subseteq SC_F(X) = S_{\lambda_1}(X) \subseteq \ldots \subseteq\ S_{\lambda^+}(X) = C(X).$$

It is also manifest that if $|X| = \lambda$, where λ is regular then $S_\lambda(X)$ is the largest proper ideal among all μ-supersocles (note, we may have $S_\lambda(X) = 0$).

Motivated by this, the next two definitions are natural and are also needed.

Definition 2.2. An element $x \in X$ is called a λ-isolated point if x has a neighborhood with cardinality less than λ. The set of λ-isolated points of X is denoted by $I_\lambda(X)$.

Definition 2.3. A space X is called λ-discrete if $I_\lambda(X) = X$.

We note that $W(\lambda)$, the space of all ordinals less than λ, where λ is a cardinal number, is a λ-discrete space, see [14, 5.11]. Clearly, a point is isolated if and only if it is $\aleph_0$-isolated, and the set of all isolated points of X is denoted by $I(X)$. We should also remind the reader that $I_{\aleph_1}(X)$ is denoted by $I_c(X)$. We should also recall here that a subspace of an $\aleph_1$-discrete space is countable if and only if it is Lindelöf, see [10]. Similarly, a subspace of a λ^+-discrete space has the cardinality λ if and only if it is λ-compact.

Evidently, every space with the cardinality λ is a λ^+-discrete space, and any finite direct product of λ-discrete spaces is λ-discrete. It also goes without saying that a subspace of a λ-discrete space is λ-discrete. Clearly, if $X = \prod_{s\in S} X_s$ is λ-discrete, then each X_i is λ-discrete too, but the converse is not necessarily true. It is also manifest that the free union $X = \bigoplus_{s\in S} X_s$ is λ-discrete if and only if each X_s is λ-discrete for each $s \in S$.

Let us recall the concept of λ-compactness in [17].

Definition 2.4. A topological space X is called λ-compact if each open cover of X can be reduced to an open cover whose cardinality is less than λ, where λ is the least infinite cardinal number with this property.

The following result is evident.

Proposition 2.5. *In a λ-discrete space, every λ-compact subspace has cardinality less than λ.*

The following lemma whose proof can be given using the proof of [13, Proposition 2.4], word for word, is needed.

Lemma 2.6. *For any space X, $I_\lambda(X) = \bigcup\{coz(f) : f \in S_\lambda(X)\}$.*

We recall that the ideal I of $C(X)$ is free if $\bigcap_{f\in I} Z(f) = \varnothing$, that is, $\bigcup\{coz(f) : f \in I\} = X$.

The following result is now immediate.

Corollary 2.7. *For any space X, the following statements hold:*

(1) *The ideal $S_\lambda(X)$ is not a zero ideal if and only if X has a λ-isolated point.*

(2) *The space X is a λ-discrete space if and only if $S_\lambda(X)$ is free.*

(3) *For each $x \in X$, $M_x = \{f \in C(X) : f(x) = 0\}$ is a maximal ideal.*

Corollary 2.8. *For any space X we have the following:*

(1) *An element x is a λ-isolated point if and only if $M_x + S_\lambda(X) = C(X)$.*

(2) *X is a λ-discrete space if and only if for all $x \in X$, $M_x + S_\lambda(X) = C(X)$.*

(3) *The ideal $S_\lambda(X)$ is a free ideal in $C(X)$ if and only if for all $x \in X$, $M_x + S_\lambda(X) = C(X)$.*

(4) *An element x is non λ-isolated point if and only if $S_\lambda(X) \subseteq M_x$.*

(5) *Let X be a topological space with $|X| \geq \lambda$ and $|I_\lambda(X)| < \lambda$. Then $S_\lambda(X) = \bigcap_{x\in X\setminus I_\lambda(X)} M_x$.*

Proof. We only give the proofs of parts (1) and (5).

(1) Let $x \in X$ be a λ-isolated point. Then by Lemma 2.6, there exists $f \in S_\lambda(X)$ such that $f(x) = 1$. So $(1-f) \in M_x$, hence $S_\lambda(X)+M_x = C(X)$. Now let $M_x + S_\lambda(X) = C(X)$. Then there exists $h \in S_\lambda(X)$ such that $(1-h) \in M_x$. This implies that $x \in X \setminus Z(h)$, where $|X \setminus Z(h)| < \lambda$. Consequently, x is a λ-isolated point.

(5) By part (4) and our assumption, $S_\lambda(X) \subseteq \bigcap_{x \notin I_\lambda(X)} M_x$. Now we may assume that $0 \neq f \in \bigcap_{x \notin I_\lambda(X)} M_x$. Hence $x \in X \setminus I_\lambda(X) \subseteq Z(f)$, and since $|I_\lambda(X)| < \lambda$, we infer that $f \in S_\lambda(X)$ and we are done. □

The following is an extension of [13, Theorem 2.7].

Theorem 2.9. *$I_\lambda(X)$ is finite if and only if $S_\lambda(X) = C_F(X)$, where $\lambda \geq \aleph_1$. In particular, in this case, $SC_F(X) = C_F(X)$.*

Proof. ($\Rightarrow$) If $I_\lambda(X)$ is finite then x is isolated for each $x \in I_\lambda(X)$. So $I_\lambda(X) = I(X)$, and consequently $S_\lambda(X) = C_F(X)$, see also [19].

($\Leftarrow$) Suppose $S_\lambda(X) = C_F(X)$ and $I_\lambda(X)$ is an infinite set, and seek a contradiction. Let $C = \{x_1, x_2, ...\} \subseteq I_\lambda(X)$ be a countable subset. Hence for each $x_n \in C$, there exists an open set G_n, with the cardinality less than λ. By completely regularity of X, for each $n \geq 1$ there exists $f_n \in C(X)$, such that $f_n(x_n) = 1$ and $f_n(X \setminus G_n) = (0)$. Now put $f = \sum_{n=1}^{\infty} \frac{f_n^2}{f_n^2+1} 2^{-n}$, and note that for each $n \geq 1$, $f(x_n) \neq 0$, and consequently $f \notin C_F(X)$, see [19]. But we claim that $f \in S_\lambda(X)$. To see this, it is enough to show that $|X \setminus Z(f)| < \lambda$. Hence it suffices to show that $X \setminus Z(f) \subseteq \bigcup_{n=1}^{\infty} G_n$. Let $x \in X \setminus Z(f)$ and $x \notin \bigcup_{n=1}^{\infty} G_n$. Thus $x \in \bigcap_{n=1}^{\infty}(X \setminus G_n)$ and $f(x) = 0$, which is the desired contradiction. □

The following proposition is evident (note that, if $\lambda > \aleph_0$ then $I_c(X) = I_{\aleph_1}(X) \subseteq I_\lambda(X)$).

Proposition 2.10. *Let $\lambda > \aleph_0$ and $|I_\lambda(X)| \leq \aleph_0$. Then $S_\lambda(X) = SC_F(X)$.*

We recite the following definition from [15].

Definition 2.11. Let (Y, τ) be an uncountable discrete space with cardinality greater than or equal to λ such that λ is regular cardinal number. Similar to the one-point compactification construction of Y, put $X = Y \cup \{x\}$, where x is not a point in Y. Let $\tau^* = \tau \cup \{G \subseteq X : x \in G, |X \setminus G| < \lambda\}$. It is clear that (X, τ^*) is a Hausdorff and λ-compact space. This space is called the one-point λ-compactification of the discrete space Y.

It is easy to check that the space X above, is completely regular. To this end, we just recall that a Hausdorff space whose set of nonisolated points is finite, is normal, see [9]. Moreover, since λ is a regular cardinal number, one can show that for any cardinal number $\gamma < \lambda$, $\bigcap_{i\in I} G_i$ is open, where every G_i is open in X and $|I| \leq \gamma$, that is, X is a P_λ-space (note, X is P_λ-space if every intersection with cardinality less than λ of open sets (that is, G_λ-set) is open). (X, τ^*) with this structure is a non λ-discrete space and $I_\lambda(X) = Y$.

The next remark gives an example of a non λ-discrete space X for which $S_\lambda(X) \neq SC_F(X)$, where λ is an infinite cardinal number.

Remark 2.12. Let $X = Y \cup \{x\}$ be the one-point λ-compactification of a discrete space Y, with $|Y| \geq \lambda \geq \aleph_1$. Let us define the map f as follows,

$$f(x) = \begin{cases} 0, & x \in G \\ 1, & x \in X \setminus G \end{cases}$$

where G is an open set containing x with $\aleph_\circ < |X \setminus G| < \lambda$. Hence $f \in S_\lambda(X) \setminus SC_F(X)$. This shows that X is a non λ-discrete space with $S_\lambda(X) \neq SC_F(X)$.

Let us recall that every λ-discrete space X is a locally λ-compact (a Hausdorff space X is called locally λ-compact if every $x \in X$ has a neighborhood which is λ-compact). An ideal I in a commutative ring R is called regular if for each $a \in I$ there exists $b \in I$ such that $a = a^2 b$. It is well-known and easy to prove that a ring R is regular if and only if there is a regular ideal I in R such that $\frac{R}{I}$ is regular, too, see [15, Lemma 1.3].

We conclude this section with the following theorem, which is our main result.

Let us recall that $O_x = \{f \in C(X) : Z(f) \text{ is a neighborhood of } x\}$, which is a fixed ideal in $C(X)$.

The following lemma, which is needed for the proof of our main theorem, generalizes the well-known fact that every countable subset of a P_λ-space is closed discrete, see [14].

Lemma 2.13. *If X is a P_λ-space, then every subset of cardinality less than λ is a closed discrete subspace.*

Proof. Let $Y \subseteq X$ with $|Y| < \lambda$ and $x \in Y$. Clearly, for each $y \in Y \setminus \{x\}$, there exists $f_y \in C(X)$ such that $f_y(x) = 0$, $f_y(y) = 1$. Therefore $x \in \bigcap_{y \in Y\setminus\{x\}} Z(f_y) = G$. Since $|Y \setminus \{x\}| < \lambda$ and X is a P_λ-space, we infer that G is an open subset of X. It goes without saying that $\{x\} = G \cap Y$ which means that $\{x\}$ is open in Y, and consequently Y is discrete. Moreover, Y is closed, by [17, Remark 1.12]. □

Theorem 2.14. *The following statements are equivalent for a space X with $|X| \geq \lambda$, where $\lambda \geq \aleph_1$ is a regular cardinal.*

(1) *X is the one-point λ-compactification of a discrete space.*

(2) *X is a P_λ-space and $S_\lambda(X) = O_x$, for some $x \in X$.*

Proof. (1) $\Rightarrow$ (2) Let $X = Y \cup \{x\}$ be the one-point λ-compactification of Y, where Y is discrete. By [17, Example 1.8], X is a P_λ-space (note, X is a P_λ-space if for any $\gamma < \lambda$, $\bigcap_{i\in I} G_i$ is open, where each G_i is open and $|I| \leq \gamma$), (we should emphasize that X is a P_λ-space too). Thus it remains to be shown that $S_\lambda(X) = O_x$. First, we prove that $S_\lambda(X) \subseteq O_x$. To see this, let $f \in S_\lambda(X)$ and note that $|X \setminus Z(f)| < \lambda$. Hence $x \notin X \setminus Z(f)$, for $|Z(f)| \geq \lambda$. Since X is a P_λ-space, we infer that $Z(f)$ is open, therefore $f \in O_x$. Now let $f \in O_x$. Then $Z(f)$ is a neighborhood of x, and therefore by definition $|X \setminus Z(f)| < \lambda$, that is, $f \in S_\lambda(X)$, and we are done.

(2) $\Rightarrow$ (1) Let $O_x = S_\lambda(X)$, for some $x \in X$, where X is a P_λ-space. Put $Y = X \setminus \{x\}$. First, we show that x is a non λ-isolated point, a fortiori nonisolated point. To see this, let G be an open set containing x whose cardinality is less than λ and get a contradiction. By complete regularity of X, there exist $f, g \in C(X)$ such that $Z(f) \cap Z(g) = \varnothing$, $x \in \operatorname{int}Z(f)$, and $X \setminus G \subseteq \operatorname{int}Z(g)$, see [14, Theorem 1.15]. Since $f \in O_x = S_\lambda(X)$, we infer that $|X \setminus Z(f)| < \lambda$. We notice that $X \setminus G \subseteq Z(g)$ and $Z(f) \subseteq X \setminus Z(g) \subseteq G$. Consequently, $|Z(f)| < \lambda$ which shows that $|X| < \lambda$, that is absurd. Now we claim that x is the only non λ-isolated point of X. If not, let $y \neq x$ be another non λ-isolated point in X and seek a contradiction. Again by complete regularity of X, there exist $f, g \in C(X)$ such that $Z(f) \cap Z(g) = \varnothing$, $x \in \operatorname{int}Z(f)$, and $y \in \operatorname{int}Z(g)$. This implies that $f \in O_x = S_\lambda(X)$ and $|X \setminus Z(f)| < \lambda$. But $Z(g) \subseteq X \setminus Z(f)$ implies that $|Z(g)| < \lambda$. So y is a λ-isolated point, which is a contradiction. Now we prove that the cardinality

of the complement of each open set containing x is less than λ. To see this, let G be an open set containing x. Then there exists $f \in C(X)$ such that $x \in Z(f) \subseteq G$ (note, $Z(f)$ is open, since X is a P_λ-space). This implies that $f \in O_x = S_\lambda(X)$, hence $|X \setminus G| \leq |X \setminus Z(f)| < \lambda$ and we are done. It remains to show that each $y \in Y$ is an isolated point, and every subset G containing x with $|X \setminus G| < \lambda$ is an open neighborhood of x. First, we show that each $y \in Y$ is an isolated point of X. We have already shown that y is a λ-isolated point in X which means that there is an open set H containing y such that $|H| < \lambda$. In view of the previous lemma, H is a discrete closed subspace of X. Since H is open, we infer that all of its points are isolated, hence y is isolated. We notice that we have already shown that the above set G, where $|X \setminus G| < \lambda$ is a neighborhood of x. □

It is interesting to note that in the proof of the part (2)⇒(1) of the previous theorem, instead of the assumption $O_x = S_\lambda(X)$, we just used the fact that $O_x \subseteq S_\lambda(X)$. As an immediate consequence of Theorem 2.14, we obtain that if in a P_λ-space X with $|X| \geq \lambda$, $O_x \subseteq S_\lambda(X)$, for some $x \in X$, then $O_x = S_\lambda(X)$ and X is the one-point λ-compactification of a discrete space Y with $|Y| \geq \lambda$. It is known that $C_F(X)$ is never a prime ideal in $C(X)$, see [10, Proposition 1.2]. In contrast to this fact, it has already been observed that $SC_F(X)$ can be a prime ideal (even a maximal ideal). Let us record this fact which is an advantage of $S_\lambda(X)$ over $C_F(X)$, where $\lambda > \aleph_0$, in the context of $C(X)$. The following corollary is proved in [13, 2.19].

Corollary 2.15. *Let X be either countable or one-point $\aleph_1$-compactification of some uncountable discrete space. Then $SC_F(X) = S_{\aleph_1}(X)$ is a prime ideal in $C(X)$.*

We immediately have the following proposition, see also [17, Example 1.8].

Proposition 2.16. *Let X be the one-point λ-compactification of a discrete space Y with $|Y| > \lambda$. Then $S_\lambda(X)$ is a prime ideal (in fact a maximal ideal) in $C(X)$.*

3 λ-Pseudo minimal ideals and λ-disjoint spaces

In this section we are trying to extend the definitions and the results of [13, Section 4] concerning the super socle of $C(X)$ to $S_\lambda(X)$. We recall that

$C_F(X)$ is a direct sum of minimal ideals in $C(X)$, which are evidently generated by idempotents. Note that, if I is a minimal ideal in $C(X)$ then $I = eC(X)$, where $e \in C(X)$ is an idempotent such that there is $x \in X$ with $e(x) = 1$ and $e(X \setminus \{x\}) = 0$ (clearly, x is an isolated point in X, see [19, Proposition 3.1]). Similarly to $C_F(X)$, in [13, lemma 4.2], it is shown that the super socle of $C(X)$ is a kind of direct sum of ideals in $C(X)$, which is not necessarily a direct sum. Motivated by this, we show that similar results hold for $S_\lambda(X)$, too.

The following definition is the counterpart of [13, Definition 4.1].

Definition 3.1. Let G be an open neighborhood of $x \in X$ with $|G| < \lambda$. Then the ideal (f^x_G), where $f^x_G \in C(X)$ such that $f^x_G(x) = 1$ and $f^x_G(X \setminus G) = (0)$ (note, by complete regularity of X, f^x_G exists and $f^x_G \in S_\lambda(X)$, but it is not necessarily unique) is called a λ-pseudo minimal ideal at x.

Let $\mathcal{L}(X)$ be the set of all open subsets of X with cardinality less than λ. Take $G \in \mathcal{L}(X)$ and $x \in G$. Then we say that an element $f \in C(X)$ is of the form f^x_G, if $f(x) = 1$ and $f(X \setminus G) = 0$. Now put

$$F^x_G = \{f \in C(X) : f \text{ is of the form } f^x_G\}$$

and $F^x_{\mathcal{L}(X)} = \bigcup_{G \in \mathcal{L}(X)} F^x_G$. Then $S_x = \sum_{f \in F^x_{\mathcal{L}(X)}} (f)$ is called the λ-pseudo socle at x. Finally, if we put

$$F^{I_\lambda(X)}_{\mathcal{L}(X)} = \{f \in C(X) : f \text{ is of the form } f^x_G, \text{where } (x, G) \in I_\lambda(X) \times \mathcal{L}(X) \text{ and } x \in G\}$$

and $S = \sum_{f \in F^{I_\lambda(X)}_{\mathcal{L}(X)}} (f)$, then $\sum_{x \in I_\lambda(X)} S_x = S \subseteq S_\lambda(X)$ is called the λ-pseudo socle of $C(X)$.

We should emphasize that (g), where $g \in C(X)$, is a λ-pseudo minimal ideal at an element $x \in I_\lambda(X)$ if and only if $g \in F^{I_\lambda(X)}_{\mathcal{L}(X)}$. We should also recall that if $x \in X$ is an isolated point then the pseudo minimal ideal $(f^x_{\{x\}})$ at x, is in fact a minimal ideal in $C(X)$, by the comment preceding the previous definition, see also [19, Proposition 3.1]. It is clear that $(f^x_{\{x\}})$ is contained in every λ-pseudo minimal ideal, (f^x_G) say, at x (note, $f^x_{\{x\}} f^x_G \neq 0$ implies that $(f^x_{\{x\}} f^x_G) = (f^x_{\{x\}}) \subseteq (f^x_G))$. Consequently,

$$(f^x_{\{x\}}) = \bigcap\{(g) : (g) \text{ is a } \lambda\text{-pseudo minimal ideal at } x\}$$

(that is, every minimal ideal in $C(X)$, which is clearly a λ-pseudo minimal ideal at an isolated point $x \in X$, is the intersection of all λ-pseudo minimal ideals at x), see also [13].

The following lemma is now evident (note, let $0 \neq f \in S_\lambda(X)$, $f(x) = r$, for some $x \in X \setminus Z(f) = G$, then $f \in (f^x_G) \subseteq S_\lambda(X) \subseteq S$, where $f^x_G = r^{-1}f$).

Lemma 3.2. $S_\lambda(X) = \sum_{x \in I_\lambda(X)} S_x = S$ *(that is, the λ-super socle and the λ-pseudo socle of $C(X)$ coincide).*

It is manifest that the above sum is not necessarily a direct sum of ideals S_x. Next, we are looking for spaces X in order to get some kind of direct sum for $S_\lambda(X)$. Let us begin with an example as a prototype.

The following example imitates [13, Example 4.3].

Example 3.3. Let X be a discrete space such that $|X| \geq \lambda$. In view of [19, Proposition 3.3] and its proof, we have $C_F(X) = \sum_{x \in X} \bigoplus (f^x_{\{x\}})$. As for the λ-super socle, first we may put $X = \bigcup\limits_{i \in I} X_i$, where each X_i is an infinite subset of X with cardinality less than λ, and for $i \neq j$, $X_i \cap X_j = \varnothing$. Now for each $i \in I$ we define the ideal $S_i = \{f \in C(X) : X \setminus Z(f) \subseteq X_i\}$. Then one easily shows that $S_\lambda(X) = \sum_{i \in I} \biguplus_\lambda S_i$, where an element f of the latter sum is of the form $f = \sum\limits_{i \in I} f_i$ with $f_i \in S_i$ such that $|\{i \in I : f_i \neq 0\}| < \lambda$, (note that the infinite sum $f = \sum\limits_{i \in I} f_i$ is well-defined, for, if $x \in X$ then $f(x) = f_i(x)$ for a unique $i \in I$). It is manifest that $\sum\limits_{i \in I} \bigoplus S_i$ exists and it is a subideal of $S_\lambda(X)$.

If $\sum\limits_{i \in I} \bigoplus S_i$ and $\sum\limits_{i \in I} \biguplus_\lambda S_i$ exist for a collection of ideals S_i in $C(X)$, then similar to [13], we call the latter sum "λ-strong direct sum" of these ideals.

Motivated by the previous example, we present the next theorem, which was promised earlier. First we need the following definition.

Definition 3.4. A space X is called λ-disjoint, if its λ-isolated points (that is, $I_\lambda(X)$) can be disjointly separated, that is, its λ-isolated points can be written as a union of disjoint collection of clopen subsets of X with cardinality less than λ.

Example 3.5. Discrete spaces with cardinality greater than or equal to λ, a topological space X without λ-isolated points (for example, $X = \mathbb{R}$ with the usual topology), the one-point λ-compactification of a discrete space D, where $|D| \geq \lambda$, the sum (that is, free union) of any collection of λ-disjoint spaces, $\mathfrak{c}$-disjoint spaces, where $\mathfrak{c}$ is the cardinality of $\mathbb{R}$, $\mathfrak{c} \leq \lambda$, and finally $X = Y \oplus Z$, where Y has no λ-isolated points and Z is a λ-disjoint space, are some examples of λ-disjoint spaces.

We conclude this section by proving that for λ-disjoint spaces, the λ-super socle of $C(X)$ is almost decomposable (that is, it is a λ-strong direct sum or a direct sum of some of its subideals).

Theorem 3.6. *Let λ be a regular cardinal number and X be a λ-disjoint space with $X = Y \cup Z$, $Y \cap Z = \varnothing$ such that $I_\lambda(X) = Z = \bigcup_{i\in I} G_i$, where each G_i is a clopen set with cardinality less than λ, and $G_i \cap G_j = \varnothing$ for all $i \neq j$. Then $S_\lambda(X) = \sum_{i\in I} \biguplus_\lambda S_i$, where $S_i = \{f \in C(X) : X \setminus Z(f) \subseteq G_i\}$. Moreover, $\sum_{i\in I} \bigoplus S_i \subseteq S_\lambda(X)$, and if I is finite then $S_\lambda(X) = \sum_{i\in I} \bigoplus S_i$.*

Proof. Let $f \in \sum_{i\in I} \biguplus_\lambda S_i$. Then $f = \sum\limits_{i\in I} f_i$ with $f_i \in S_i$. This implies that $X \setminus Z(f) \subseteq \bigcup\limits_{i\in J} X \setminus Z(f_i)$, where $J \subseteq I$ with $|J| < \lambda$ such that $f_i \neq 0$ for all $i \in J$. But for each i, $X \setminus Z(f_i) \subseteq G_i$ implies $|X \setminus Z(f_i)| < \lambda$, and consequently $|X \setminus Z(f)| < \lambda$. Hence $f \in S_\lambda(X)$. Conversely, let $f \in S_\lambda(X)$. Now for each $i \in I$, we define $f_i \in C(X)$ by $f_i(x) = f(x)$ for each $x \in G_i$ and $X \setminus G_i \subseteq Z(f_i)$. So $X \setminus Z(f_i) \subseteq G_i$, and therefore $f_i \in S_i$, for all $i \in I$. Since $|X \setminus Z(f)| < \lambda$, we infer that $X \setminus Z(f) \subseteq Z$ which implies that $f(G_i) \neq 0$, where $|\{i \in I : f(G_i) \neq 0\}| < \lambda$. Clearly, $f(Y) = 0$. Thus, whenever $f(x) \neq 0$, there is a unique $i \in I$ with $x \in G_i$ such that $f_i(x) = f(x)$, which immediately shows that $f = \sum\limits_{i\in I} f_i$, where $f_i \in S_i$, and we are done. The last part is evident. □

Definition 3.7. Let $\{A_i : i \in I\}$ be a collection of ideals in $C(X)$. If for each $f_i \in A_i$, $\sum\limits_{i\in I} f_i \in \mathbb{R}^X$, where $(\sum\limits_{i\in I} f_i)(x) = \sum\limits_{i\in I} f_i(x)$ is well-defined for all

$x \in X$, then by the external sum of these ideals, we mean

$$\sum_{i\in I}^{ex} A_i = \{f \in \mathbb{R}^X : f = \sum_{i\in I} f_i, f_i \in A_i\}.$$

Clearly, $\sum\limits_{i\in I}^{ex} A_i$ may not be an ideal in $C(X)$ (note, it is indeed an ideal in $C(X)$ if I is finite), but it is naturally a $C(X)$-module.

Remark 3.8. Let $X = \oplus_{i\in I} X_i$ be the sum (free union) of spaces X_i, where $|X_i| \geq \lambda$, for each $i \in I$ and $I(X)$ be the set of isolated points of X. For each $i \in I$, define $e_i(X_i) = 1$ and $e_i(X_j) = 0$ for all $j \neq i$. Then we may assume that $C(X_i) = e_i C(X)$. Clearly, $\sum\limits_{i\in I} \oplus C(X_i) \subseteq C(X) = \sum\limits_{i\in I}^{ex} C(X_i)$ (note, $1 = \sum\limits_{i\in I} e_i$). In view of [19, Propositions 3.1, 3.3], one can easily see that $C_F(X) = \sum\limits_{x\in I(X)} \oplus(f^x_{\{x\}}) = \sum\limits_{i\in I} C_F(X_i)$. Moreover, $S_\lambda(X) = \sum\limits_{i\in I} \biguplus_\lambda S_\lambda(X_i)$. Let us prove the latter equality. Let $f \in S_\lambda(X)$. Then $|X \setminus Z(f)| < \lambda$, hence $|J| < \lambda$, where $J = \{i \in I : X_i \cap (X \setminus Z(f)) \neq \varnothing\}$. For each $i \in J$ put $G_i = X_i \cap (X \setminus Z(f))$. Now define $f_i = e_i f$, for each $i \in I$ and note that $f(G_i) \neq 0$, whenever $i \in J$. Clearly, $f_i \in S_\lambda(X_i)$ for each $i \in I$, hence $f = \sum\limits_{i\in I} f_i \in \sum\limits_{i\in I} \uplus S_\lambda(X_i)$. The converse is evident.

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