Categories and
General Algebraic Structures
with Applications
Volume 4, Number 1, February 2016, 1-7

Birkhoff's Theorem from a geometric perspective: A simple example

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Abstract. From Hilbert's theorem of zeroes, and from Noether's ideal theory, Birkhoff [1] derived certain algebraic concepts (as explained by Tholen [10]) that have a dual significance in general toposes, similar to their role in the original examples of algebraic geometry. I will describe a simple example that illustrates some of the aspects of this relationship.

The dualization from algebra to geometry in the basic Grothendieck spirit can be accomplished (without intervention of topological spaces) by the following method, known as Isbell conjugacy [3], [5].

1 Isbell Conjugacy

Any given small category can be used as the category of figure shapes, within the larger category of contravariant set-valued functors on it, which we call the category of pre-spaces. This full embedding is achieved by the basic Yoneda construction. The general pre-space consists of figures of the various shapes, which have been called 'generalized elements' by Kock, 'elements' by Volterra, and 'points' by Grothendieck; however, we will reserve

Keywords: Grothendieck spectrum, Cantor, Boole, Hilbert, Birkhoff: Existence and sufficiency of generalized points, reflexive graphs.
Mathematics Subject Classification [2010]: 03C05, 03Exx, 05Cxx, 58Axx.
Received: 15 November 2015 (Invited Paper)
ISSN Print: 2345-5853 Online: 2345-5861

the term ‘points’ for figures whose shapes are co-simple objects in the given small category. A crucial role in a generalized Birkhoff concept will be played by a category intermediate between figures and points, consisting of generalized points in a sense suggested by the Leibniz notion of monads. The morphisms of the given small category operate (on the ‘right’) on these possibly singular figures, permitting the definition via divisibility of incidence relations (intersection, et cetera). Those incidence relations are preserved by the natural transformations that are the morphisms of pre-spaces; these morphisms can therefore be thought of as generalized continuous maps. The word ‘space’ can be reserved for the objects of an appropriate reflective sub-category of the pre-spaces; if the reflection is left exact, the spaces will also form a topos.

The other Yoneda embedding contravariantly assigns to each object of the original category a covariant set-valued functor (a representable pre-algebra that provides function types for all the ‘pre-algebras’). The morphisms in the original small category act as ‘left’ algebraic operations on the functions, so that the natural transformations act as homomorphisms in the sense that they commute with these operations. Again, the consideration of a reflective sub-category is sometimes appropriate (with the word ‘algebra’ reserved for its objects).

The pre-spaces and the pre-algebras are connected by adjoint functors called ‘spectrum’ and ‘function algebra’, defined by naturality as follows:

$$Spec(A)(C) = Nat(A, C^{op}), \qquad F(X)(C) = Nat_{op}(X, C)$$

for each pre-algebra A and each pre-space X, where C and C^{op} result from the two Yoneda embeddings. The two Yoneda embeddings are transformed into each other by these ‘conjugate’ adjoints. In case the given small category has finite products, it is often reasonable to define ‘algebras’ to be those pre-algebras that actually preserve the products; all the function algebras coming from pre-spaces will be algebras in that sense. On the other hand, if the small category has coproducts that are extensive [7] , then the pre-spaces that take those co-products to products may reasonably be called ‘spaces’; the spectrum of any pre-algebra is a space. Thus the two adjoint functors *Spec* and F typically restrict to an adjoint pair connecting such algebras and such spaces; this adjointness is considered to be the basic relation of algebraic geometry. It should not be expected that this adjointness be an equivalence of categories, but it provides useful invariants of spaces and of algebras.

2 Points and 'Zeroes'

Of particular interest are figures whose shape is a 'co-simple' object of the given small category; such figures are called points. Birkhoff pointed out the importance of an intermediate category of shapes more general than points. In the case of commutative algebra his intermediate category includes the spectra of those special local rings called sub-directly irreducible. Their special virtue is that they lead to a faithful representation of more general algebras, in terms of a striking picture that I will explain. In a general setting I will discuss a partial analog of this intermediate category.

Hilbert's theorem showed that every non-trivial space has at least one point (i.e. a figure whose shape is co-simple). The expression 'zeroes' for these points derives from the special role of subtraction in the particular algebras known as commutative rings. The basic content is the existence of solutions to equations which require that two functions take the same value. Of course, when subtraction exists, such solutions are just points at which the difference of the two functions vanishes, i.e. Nullstellen for that difference function. (This sort of reduction can be achieved without subtraction in any category with sufficient exactness properties: in each object of the form $Y \times Y$ we can collapse the diagonal Y to a single point that we could call 'zero'. Then the solutions of an equation involving a pair of maps from X to Y will constitute the same subspace of X as the one where the single map, obtained by following their paired map by the quotient map, takes on the single value '0'.)

3 The Leibniz Core and the Birkhoff Property of an Object

The Noether-Birkhoff theorem is stronger than Hilbert's theorem: By considering a small generalization of 'co-simple', one obtains not merely the existence of points, but the sufficiency of the resulting notion of 'generalized points'; here sufficiency refers to the capacity to separate functions. In terms of algebras sufficiency means that a certain induced homomorphism to a product of very special algebras will always be monomorphic. The geometric way to guarantee such a monomorphic map of algebras involves an induced ‘pseudo-epimorphism’ from an amalgam of special ‘tiny’ spaces. Here ‘pseudo’ refers to the ‘perception’ powers of a space R that represents functions on X as $X \to R$. In the classical case R itself is a representable figure shape serving as a function type. These functions constitute the algebras that we aim to faithfully represent. *The Leibniz Core of a space X is the union $L(X)$ of all its generalized points; this is obtained as the right adjoint of the inclusion functor from the subcategory of those spaces that ‘look like clouds of Leibnizian monads’. The more general figures that substantiate cohesion between points are omitted in the reduction from X to $L(X)$, but each point may have self-cohesion (which is retained in $L(X)$).*

In algebraic geometry, analytic geometry, and smooth geometry such a sub-category has been usefully explained with the help of nilpotent quantities, but there is a more qualitative characterization of a suitable such intermediate category of ‘Leibnizian’ spaces: every connected component of a Leibnizian space has exactly one point. (This exemplifies the notion of intensive quality in the axiomatic theory of cohesion.) [6]

There are several situations where such a category intermediate between point-like spaces and general spaces plays a significant role in analyzing the cohesion and motion within the general spaces. The Leibniz picture of a cloud of monads can be more concretely realized, for example in the study of singularities of smooth maps; the self-cohesion of each point where a given smooth function vanishes can be expressed by higher-order differential information, and the topos of all smooth spaces allows for subspaces (typically the spectra of algebras that are linearly finite-dimensional) where everything except this infinitesimal information is omitted.

4 Boole, Cantor, and Discreteness

In any topos there is the sub-topos of all the spaces satisfying the sheaf property with respect to the double-negation modal operator on the truth-value space. This Boolean subtopos in some very special cases is ‘essential’ in the sense that the associated sheaf functor has a further left adjoint [8]. That left adjoint we will call the inclusion of discrete spaces. This gives rise to a comonad that we might call the Cantor comonad, because it captures exactly the first step of Cantor’s set theory: to extract from any

Menge its subspace of 'lauter Einsen' (extending the ancient Greek notion of arithmos to infinite spaces) [2]. In case this comonad is left exact, it expresses the Boolean topos of sheaves alternatively as a quotient topos, with the quotient functor being the one assigning to the Mengen (or spaces) their underlying abstract sets in a relativization of the sense that is used in modern mathematics. (The abstract sets implicit in the small categories and set functors occurring in the above conjugate construction of examples can also be understood in this more precise Boole-Cantor way.)

These two properties (existence and left-exactness of the Cantor comonad) can be further strengthened to permit a simple description of the desired intermediate category. Namely, a *connected components* functor is a still further left adjoint that moreover preserves finite products. The adjunction morphisms that accompany the functors in the resulting quartet can be composed to yield, in particular, a natural transformation from the Cantor points of any space to the discrete space of connected components of that same space; the epimorphicity of this map would express Hilbert's theorem of zeroes, in the sense that in each component the required points would exist at least after passing to a covering (involving field extensions in the classical case). However, here we use a possible stronger property of this natural map to single out the intermediate subcategory of interest.

Definition 4.1. The Leibniz spaces are just those spaces for which the natural map from points to components is a bijection. A Birkhoff space R is a space having the uniqueness property with respect to the Leibniz core functor: for any X, any 'infinitesimal' map $L(X) \to R$ can be integrated in at most one way to a global function $X \to R$.

Remark 4.2. The Leibniz core of a space X can be characterized as the largest sub-space of X that retracts onto the Cantor core of X.

5 The Example of Reflexive Graphs

We now apply the above definitions to the ordinary topos of reflexive directed (multi)graphs [9]. These are pre-spaces whose figure types are described by a small category with five non-identity maps and two objects, one of which is terminal. The corresponding 'pre-algebras' can be pictured as cylinders, each of which has a graph as its spectrum [4].

Proposition 5.1.
(BC) *The Boolean subtopos consists of those graphs with the property that between any two points there is a unique arrow, whereas the Cantor subcategory consists of those graphs that have no non-identity arrows; the obvious fact that these two totally different subcategories are equivalent to one another as categories* (*ignoring the inclusion functors*) *expresses a simple example of functorial Unity and Identity of Opposites* (*UIO*) [4].
(L) *The Leibniz graphs are those consisting entirely of loops, so that the inclusion* $L(X) \to X$ *omits those arrows from* X *whose source and target are distinct.*
(B) *The 'Birkhoff graphs', namely those* R *for which a map* $X \to R$ *is uniquely determined by its restriction to* $L(X)$*, are characterized by the property that*

$$\forall a, b,\ \forall x, y : a \to b \Rightarrow x = y.$$

The usual name for such a graph is reflexive binary relation.

In the general context discussed above the subcategory of Birkhoff spaces will be closed under subobjects as well as under products (and indeed every space will have a largest Birkhoff quotient), and this can be directly verified in the example of graphs. Hence there are many non-representable spaces that are Birkhoff spaces, but the following example shows that not all spaces in the topos are Birkhoff. The truth-value graph is not a Birkhoff graph and indeed in any topos where the inclusion map from $L(X) \to X$ is monic but not epic for some X, the truth-value object cannot have the Birkhoff property, because the constantly-true map has the same restriction as the classifying map for this subobject, yet they are not equal.

I thank Francisco Marmolejo and Matias Menni for enlightening discussions concerning the topics treated in this note.

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