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The prime state ideal theorem in state residuated lattices

Francis Woumfo*, Etienne Romuald Temgoua Alomo, Celestin Lele

Abstract. The aim of this paper is to establish the prime state ideal theorem in state residuated lattices (SRLs). We study the state ideals lattice $\mathcal{SI}(L)$ of a state residuated lattice (L, φ) and prove that it is a complete Brouwerian lattice in which the meet and the join of any two compact elements are compact (coherent frame). We characterize the notion of prime state ideals in SRLs. In addition, we establish the condition for which the lattice $\mathcal{SI}(L)$ is a Boolean algebra.

1 Introduction

The origin of residuated lattices is in mathematical logic without contraction. Apart from their logical interest, residuated lattices have important algebraic properties as it is well known that the algebraic study of logical systems plays an important role and have considerable applications in artificial intelligence.

The notion of state emerging from the theory of quantum mechanics was

* Corresponding author
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firstly applied to MV-algebras by Kôpka and Chovanec in [23] and then extended to non-commutative MV-algebras in [7]. Since then, the theory of states has been applied to other algebras such as (pseudo) BL-algebras ([12]), (non-commutative) Rℓ-monoids ([8]), (non-commutative) residuated lattices ([21]). By extending the codomain of a state to more general algebraic structures in order to provide an algebraic foundation for probabilities of fuzzy events inside Łukasiewicz infinite-valued logic, a new approach to states on MV-algebras was introduced by Flaminio and Montagna in [10, 11], where they added a unary operator φ to the language of MV-algebras as an internal state (or a state operator), which preserves the usual properties of states. They showed some fundamental results about state MV-algebras. Consequently, the concept of state operators has become a prominent research field in the theory of fuzzy logics and algebras [3, 5, 6, 9, 15, 22].
In 2018, using the De Morgan property (DMP): $(x \wedge y)' = x' \vee y'$, Liviu-Constantin Holdon introduced an important variety of residuated lattices called De Morgan residuated lattice which comprises salient subclasses of residuated lattices such as Boolean algebras, BL-algebras, MTL-algebras, MV-algebras, IMTL-algebras, Stonean residuated lattices and regular residuated lattices (see [17]). Recently in 2022, F. Woumfo et al, [29] studied the lattice of all state ideals of a De Morgan state residuated lattice and proved the prime state ideals theorem. It is worth nothing that, the condition (DMP) has simplified many calculations. However, similar results on algebras without (DMP) are missing so far. This paper seeks to extend our research in the more general class of state residuated lattices. The prime state ideal theorem is established. Moreover, we prove that the state ideals lattice $\mathcal{SI}(L)$ of a state residuated lattice (L, φ) is a coherent frame and we characterize the SRL for which the lattice $\mathcal{SI}(L)$ is a Boolean algebra.

This work is organized into three sections: in the first one, we present some preliminaries comprising the basic definitions, some rules of calculus and theorems that are needed in the sequel. Section 2 studies the algebraic structure of the set $\mathcal{SI}(L)$ of all state ideals in a SRL (L, φ). It is shown that $(\mathcal{SI}(L), \subseteq)$ is a coherent frame. Also, we characterize the SRL for which the lattice $\mathcal{SI}(L)$ is a Boolean algebra. In Section 3, we put emphasis on the prime state ideals by characterizing them, and prove the prime state ideal theorem.

2 Preliminaries

We summarize here some fundamental definitions and results about residuated lattices. For more details, we refer the reader to the papers [4, 24, 26, 28].

Definition 2.1. [4] A nonempty set L with four binary operations $\wedge, \vee, \odot, \to$ and two constants $0, 1$ is called a *bounded integral commutative residuated lattice* or shortly *residuated lattice* if the following properties are verified:

(C1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice;

(C2) $(L, \odot, 1)$ is a commutative monoid (with the unit element 1);

(C3) For all $x, y \in L$, $x \odot y \leq z$ iff $x \leq y \to z$.

Definition 2.2. [17] A residuated lattice satisfying the *De Morgan property* (DMP): $(x \wedge y)' = x' \vee y'$ is called a *De Morgan residuated lattice.*

The following notations of residuated lattices will be used:

Note 2.3. L will stand for a residuated lattice $(L, \wedge, \vee, \odot, \to, 0, 1)$. For any $x \in L$ and $n \in \mathbb{N}^*$, $x' := x \to 0$, $x'' := (x')'$, $x^0 := 1$ and $x^n := x^{n-1} \odot x$.

The following basic arithmetic of residuated lattices will be used.

Lemma 2.4. [4, 24]
In any residuated lattice L, the following hold for any $x, y, z \in L$:

(RL1) $1 \to x = x$, $x \to x = 1$, $x \to 1 = 1$, $0 \to x = 1$;

(RL2) $x \leq y \Leftrightarrow x \to y = 1$;

(RL3) $x \to y = y \to x = 1 \Leftrightarrow x = y$;

(RL4) *if* $x \leq y$, *then* $y \to z \leq x \to z$, $z \to x \leq z \to y$, $x \odot z \leq y \odot z$ *and* $y' \leq x'$;

(RL5) $x \odot (x \to y) \leq y$; $x \odot (x \to y) \leq x \wedge y$;

(RL6) $x \odot y \leq x \wedge y \leq x, y \leq x \vee y$; $x \leq y \to x$; $x \odot y \leq x \to y, y \to x$;

(RL7) $(x \odot y)'' = x'' \odot y'', (x \vee y)' = x' \wedge y'$ *and* $(x \wedge y)' \geq x' \vee y'$;

(RL8) $0' = 1$, $1' = 0$;

(RL9) $x \leq x'' \leq x' \to x$;

(RL10) $x \to y \leq y' \to x'$;

(RL11) $x''' = x'$, $(x \odot y)' = x \to y' = y \to x' = x'' \to y'$;

(RL12) $x \odot x' = 0$, $x \odot y = 0 \Leftrightarrow x \leq y'$; $x \odot 0 = 0$;

(RL13) $x' \to y \leq (x' \odot y')'$, $x' \odot y' \leq (x' \to y)'$, $x' \odot y' \leq (x \odot y)'$;

(RL14) $x \to (x \wedge y) = x \to y$;

(RL15) $x \odot y = x \odot (x \to x \odot y)$;

(RL16) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$, $x \odot (y \wedge z) \leq (x \odot y) \wedge (x \odot z)$, $x \vee (y \odot z) \geq (x \vee y) \odot (x \vee z)$.

Let L be a residuated lattice. We set $x \oplus y = (x' \odot y')'$, for every $x, y \in L$. Here are some properties of the operation $\oplus$ (see [2, 26, 29]).

Lemma 2.5. [2] *Let L be a residuated lattice. For any $x, y, z, t \in L$, we have:*

(P1) $x \oplus y = x' \to y'' = y' \to x''$;

(P2) $x \oplus x' = 1$, $x \oplus 0 = x''$, $x \oplus 1 = 1$;

(P3) $x \oplus y = y \oplus x$, $x, y \leq x \oplus y$;

(P4) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(P5) *If $x \leq y$, then $x \oplus z \leq y \oplus z$;*

(P6) *If $x \leq y$ and $z \leq t$, then $x \oplus z \leq y \oplus t$.*

For any $x \in L$ and $n \in \mathbb{N}$, we define $0x = 0$, $1x = x$ and $nx = (n-1)x \oplus x$, for $n \geq 2$.

Lemma 2.6. [26, 29] *The following hold for any $x, y \in L$ and $m, n \in \mathbb{N}^*$:*

(P7) $m \leq n \Rightarrow mx \leq nx$. *In particular,* $x \leq nx$;

(P8) $x \leq y \Rightarrow mx \leq my$;

(P9) $n(x \oplus y) = nx \oplus ny$;

(P10) $x \oplus ny \leq n(x \oplus y)$;

(P11) $x, y \leq x \oplus y$;

(P12) $[(x')^n]' = nx$;

(P13) $(x \oplus y)'' = x \oplus y = x'' \oplus y''$;

(P14) $x \wedge (y_1 \oplus ... \oplus y_n) \leq (x'' \wedge y_1'') \oplus ... \oplus (x'' \wedge y_n'')$.

Definition 2.7. [17] A nonempty subset I of a residuated lattice L is called an **ideal** if the following conditions are satisfied for every $x, y \in L$:

(I1) if $y \in I$ and $x \leq y$, then $x \in I$;

(I2) if $x, y \in I$, then $x \oplus y \in I$.

The set of all ideals of a residuated lattice L will be denoted by $\mathcal{I}(L)$.

Remark 2.8. It is easy to see that for all $I \in \mathcal{I}(L)$, $0 \in I$; $x \in I$ if and only if $x'' \in I$, for any $x \in L$.

Now, we give some necessary results for the sequel about lattices and frames. It is worth noting that the main references for frame theory are the following books (see [18, 25]).

Definition 2.9. [15] A lattice $(L, \wedge, \vee)$ is called *Brouwerian* if it satisfies the equality $x \wedge (\bigvee_{k \in K} y_k) = \bigvee_{k \in K} (x \wedge y_k)$ (whenever the arbitrary joins exist), for any $x, y_k \in L$, $k \in K$.

Definition 2.10. [13] We call *frame* a complete lattice L that satisfies the infinite distributive law $x \wedge \bigvee A = \bigvee \{x \wedge a : a \in A\}$, for all $x \in L$ and $A \subseteq L$.

Remark 2.11.
1. Every Brouwerian lattice $(L, \wedge, \vee)$ is distributive;
2. A frame is a complete Brouwerian lattice.

From [15, 20], an element a of a complete lattice L is called *compact* if for all $A \subseteq L$, $a \leq \bigvee A$ implies that $a \leq \bigvee H$ for some finite $H \subseteq A$.

We will denote by $\mathcal{C}(L)$ the set of all compact elements of a complete lattice L.

Proposition 2.12. [30] *Let L be a frame and $x \in L$. Then, $x \in \mathcal{C}(L)$ if for all $A \subseteq L$, $x = \bigvee A$ implies that $x = \bigvee H$ for some finite $H \subseteq A$.*

Definition 2.13. [30] A frame L is called *coherent* if the following conditions hold:

(i) $\mathcal{C}(L)$ is a sublattice of L, i.e., for all $x, y \in L$, if $x, y \in \mathcal{C}(L)$, then $x \wedge y, x \vee y \in \mathcal{C}(L)$;

(ii) For all $x \in L$, $x = \bigvee_{k \in K} x_k$, *with* $x_k \in \mathcal{C}(L)$.

In other words, a coherent frame is a complete Brouwerian algebraic lattice in which the meet and the join of any two compact elements are compact.

Definition 2.14. [14] Let L be a lattice with 0 and $x \in L$. Then $y \in L$ is said to be a *pseudocomplement* of x if $x \wedge y = 0$ and for every $z \in L$, $x \wedge z = 0$ implies $z \leq y$. L is called *pseudocomplemented* if every element has a pseudocomplement.
For every $a, b \in L$, we call a *relative pseudocomplement* of a with respect to b, the greatest element (if it exists) $c \in L$ such that $a \wedge c \leq b$.

Proposition 2.15. [14] *Every frame is pseudocomplemented.*

The concepts of state operators and state residuated lattices were introduced in 2015 by Pengfei He et al. in [15].

Definition 2.16. [15, 29] A map $\varphi : L \to L$ is said to be a *state operator* on L if the following conditions hold for any $x, y \in L$:

(SO1) $\varphi(0) = 0$;

(SO2) $x \to y = 1$ implies $\varphi(x) \to \varphi(y) = 1$;

(SO3) $\varphi(x \to y) = \varphi(x) \to \varphi(x \wedge y)$;

(SO4) $\varphi(x \odot y) = \varphi(x) \odot \varphi(x \to (x \odot y))$;

(SO5) $\varphi(\varphi(x) \odot \varphi(y)) = \varphi(x) \odot \varphi(y)$;

(SO6) $\varphi(\varphi(x) \to \varphi(y)) = \varphi(x) \to \varphi(y)$;

(SO7) $\varphi(\varphi(x) \vee \varphi(y)) = \varphi(x) \vee \varphi(y)$;

(SO8) $\varphi(\varphi(x) \wedge \varphi(y)) = \varphi(x) \wedge \varphi(y)$.

The pair (L, φ) is said to be a *state residuated lattice*, or more precisely, a *residuated lattice with internal state.*
The *kernel* of τ is the set $ker(\tau) := \{x \in L : \tau(x) = 1\}$. Analogously, the *co-kernel* of τ is the set $coker(\tau) := \{x \in L : \tau(x) = 0\}$.

From now on, unless othewise specified, (L, φ) will always denote a state residuated lattice $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$, that is, L is a residuated lattice and φ is a state operator on L.

Definition 2.17. [29] An ideal I of L is said to be a *state ideal* of (L, φ) if $\varphi(I) \subseteq I$, (i.e., for all $x \in L$, $x \in I \Rightarrow \varphi(x) \in I$).

$\mathcal{SI}(L)$ will stand for the set of all state ideals of (L, φ). It is obvious that $\{0\}, L \in \mathcal{SI}(L) \subseteq \mathcal{I}(L)$.

For computational issues, we will use the following properties.

Lemma 2.18. [15, 29] *For any $x, y \in L$, for all $n \geq 1$, we have:*

(SO9) $\varphi(1) = 1$;

(SO10) $x \leq y$ *implies* $\varphi(x) \leq \varphi(y)$;

(SO11) $\varphi(x') = (\varphi(x))'$;

(SO12) $\varphi(x \odot y) \geq \varphi(x) \odot \varphi(y)$ *and if* $x \odot y = 0$, *then* $\varphi(x \odot y) = \varphi(x) \odot \varphi(y) = 0$;

(SO13) *If* $x \leq y$, *then* $\varphi(x \odot y') = \varphi(x) \odot (\varphi(y))'$;

(SO14) $\varphi(x \rightarrow y) \leq \varphi(x) \rightarrow \varphi(y)$. *Particulary, if* x, y *are comparable, then* $\varphi(x \rightarrow y) = \varphi(x) \rightarrow \varphi(y)$;

(SO15) *If* φ *is faithful, then* $x < y$ *implies* $\varphi(x) < \varphi(y)$;

(SO16) $\varphi^2(x) = \varphi(\varphi(x)) = \varphi(x)$;

(SO17) $\varphi(L) = Fix(\varphi)$, *where* $Fix(\varphi) = \{x \in L : \varphi(x) = x\}$;

(SO18) $\varphi(L)$ *is a subalgebra of* L;

(SO19) $ker(\varphi)$ *is a state filter of* (L, φ);

(SO20) $coker(\varphi)$ *is a state ideal of* (L, φ);

(SO21) $(\varphi(x))'' = \varphi(x'')$;

(SO22) $\varphi(x \oplus y) \leq \varphi(x) \oplus \varphi(y)$;

(SO23) *If* $x, y \in \varphi(L)$, *then* $x \oplus y \in \varphi(L)$;

(SO24) $\varphi(nx) \leq n\varphi(x)$.

Remark 2.19. [15] (L, id_L) is a state residuated lattice. That is a residuated lattice L can be view as a state residuated lattice. One can see that, each ideal of L is a state ideal of (L, id_L).

Note 2.20. For any nonempty subset X of L, we denote by $\langle X\rangle_\varphi$ the state ideal of (L,φ) generated by X, that is, $\langle X\rangle_\varphi$ is the smallest state ideal of (L,φ) containing X and for an element $a \in L$, $\langle a\rangle_\varphi := \langle\{a\}\rangle_\varphi$ is called the *principal state ideal* of (L,φ). If $I \in \mathcal{SI}(L)$ and $a \notin I$, we denote by $\langle I, a\rangle_\varphi := \langle I \cup \{a\}\rangle_\varphi$.

Remark 2.21. By definition, we have $\langle\emptyset\rangle_\varphi = \{0\}$ and $\langle I\rangle_\varphi = I$, for any state ideal I of (L,φ).

The next theorem gives the concrete description of the state ideal generated by a nonempty subset of a state residuated lattice (L,φ).

Theorem 2.22. [29]
Let X be a nonempty subset of L, $I, I_1, I_2 \in \mathcal{SI}(L)$ and $a \in L\backslash I$. Then:

(1) $\langle X\rangle_\varphi = \{x \in L : x \leq n_1(x_1 \oplus \varphi(x_1)) \oplus ... \oplus n_k(x_k \oplus \varphi(x_k)),\ for\ some\ k \in \mathbb{N}^*,\ x_i \in X,\ n_i \in \mathbb{N}^*,\ for\ 1 \leq i \leq k\}$;

(2) $\langle a\rangle_\varphi = \{x \in L : x \leq n(a \oplus \varphi(a)), for\ some\ n \geq 1\}$;

(3) $\langle I, a\rangle_\varphi = \{x \in L : x \leq i \oplus n(a \oplus \varphi(a)), for\ some\ i \in I\ and\ n \geq 1\}$;

(4) $I_1 \vee I_2 := \langle I_1 \cup I_2\rangle_\varphi = \{x \in L : x \leq i_1 \oplus i_2, with\ i_1 \in I_1\ and\ i_2 \in I_2\}$.

Lemma 2.23. [29]
For all $a, b \in L$, we have:

(5) $a \leq b \Rightarrow \langle a\rangle_\varphi \subseteq \langle b\rangle_\varphi$;

(6) $\langle\varphi(a)\rangle_\varphi \subseteq \langle a\rangle_\varphi$;

(7) $\langle a \oplus \varphi(a)\rangle_\varphi = \langle a\rangle_\varphi$;

(8) $\langle(a \oplus \varphi(a)) \wedge (b \oplus \varphi(b))\rangle_\varphi \subseteq \langle a\rangle_\varphi \cap \langle b\rangle_\varphi$;

(9) $\langle a\rangle_\varphi \vee \langle b\rangle_\varphi = \langle a \vee b\rangle_\varphi = \langle a \oplus b\rangle_\varphi$.

Proposition 2.24. [29] *$(\mathcal{SI}(L), \subseteq)$ is a bounded complete lattice with the bottom element $\{0\}$ and the top element L.*

3 The lattice of state ideals of a state residuated lattice

In this section, we focus on the algebraic structure of the set $\mathcal{SI}(L)$ of all state ideals of a state residuated lattice (L, φ).

Proposition 3.1. *$(\mathcal{SI}(L), \subseteq)$ is a Brouwerian lattice.*

Proof. Let K be an index set, $I \in \mathcal{SI}(L)$, and $\{I_k\}_{k\in K}$ be a family of state ideals of (L, φ). We will show that $I \wedge (\bigvee_{k\in K} I_k) = \bigvee_{k\in K} (I \wedge I_k)$. That is, $I \cap \langle \bigcup_{k\in K} I_k\rangle_\varphi = \langle \bigcup_{k\in K} (I \cap I_k)\rangle_\varphi$. Clearly, $\langle \bigcup_{k\in K} (I \cap I_k)\rangle_\varphi \subseteq I \cap \langle \bigcup_{k\in K} I_k\rangle_\varphi$.

Let $x \in I \cap \langle \bigcup_{k\in K} I_k\rangle_\varphi$. Then $x \in I$ and $x \in \langle \bigcup_{k\in K} I_k\rangle_\varphi$. It follows that there exist $k_1, k_2, ..., k_m \in K$, $x_{k_j} \in I_{k_j}$, $1 \leq j \leq m$, such that $x \leq x_{k_1} \oplus x_{k_2} \oplus .... \oplus x_{k_m}$. Then, $x = x \wedge (x_{k_1} \oplus x_{k_2} \oplus .... \oplus x_{k_m}) \overset{(P14)}{\leq} (x'' \wedge x''_{k_1}) \oplus (x'' \wedge x''_{k_2}) \oplus .... \oplus (x'' \wedge x''_{k_m})$. Since $I, I_{k_j} \in \mathcal{SI}(L)$, we have by Remark 2.8 that $x'' \wedge x''_{k_j} \in I \cap I_{k_j}$, for every $1 \leq j \leq m$. We deduce that $x \in \langle \bigcup_{k\in K} (I \cap I_k)\rangle_\varphi$. Hence, $I \cap \langle \bigcup_{k\in K} I_k\rangle_\varphi \subseteq \langle \bigcup_{k\in K} (I \cap I_k)\rangle_\varphi$, that is $I \wedge (\bigvee_{k\in K} I_k) = \bigvee_{k\in K} (I \wedge I_k)$. Therefore, $(\mathcal{SI}(L), \subseteq)$ is a Brouwerian lattice. □

Theorem 3.2. *The lattice $(\mathcal{SI}(L), \subseteq)$ is a frame.*

Proof. From Proposition 2.24, $(\mathcal{SI}(L), \subseteq)$ is a complete lattice. From Proposition 3.1, $(\mathcal{SI}(L), \subseteq)$ is a Brouwerian lattice. Combining them, we have by Remark 2.11 (2) that $(\mathcal{SI}(L), \subseteq)$ is a frame. □

In the following result, we describe the right adjoint of the map,

$$\begin{array}{rcl} I_1 \cap \_ : \mathcal{SI}(L) & \longrightarrow & \mathcal{SI}(L) \\ I & \longmapsto & (I_1 \cap \_)(I) = I_1 \cap I. \end{array}$$

Now, for any $I_1, I_2 \in \mathcal{SI}(L)$, we put $I_1 \to I_2 = \{x \in L : I_1 \cap \langle x\rangle_\varphi \subseteq I_2\}$.

Theorem 3.3. *In the frame $(\mathcal{SI}(L), \subseteq)$, for any $I_1, I_2, I \in \mathcal{SI}(L)$, we have:*

1. $I_1 \to I_2 \in \mathcal{SI}(L)$;

2. $I_1 \cap I \subseteq I_2 \Leftrightarrow I \subseteq I_1 \to I_2$, *that is,*

$$I_1 \to I_2 = sup\{I \in \mathcal{SI}(L) : I_1 \cap I \subseteq I_2\}$$

and

$$\begin{array}{rcl} I_1 \to_{\_} : \mathcal{SI}(L) & \longrightarrow & \mathcal{SI}(L) \\ I & \longmapsto & (I_1 \to_{\_})(I) = I_1 \to I \end{array}$$

is the right adjoint of $I_1 \cap_{\_} : \mathcal{SI}(L) \longrightarrow \mathcal{SI}(L)$;

3. $I_1 \to I_2 = \{x \in L : i \wedge n(x \oplus \varphi(x)) \in I_2,\ for\ all\ i \in I_1\ and\ n \in \mathbb{N}^*\}$.

Proof. (1) We will show that $I_1 \to I_2$ is a state ideal of (L, φ). Clearly, $I_1 \to I_2 \neq \emptyset$. In fact, $\langle 0 \rangle_\varphi = \{0\}$ and $I_1 \cap \langle 0 \rangle_\varphi = \{0\} \subseteq I_2$. Hence $0 \in I_1 \to I_2$.

Now, let $x, y \in I_1 \to I_2$. Then $I_1 \cap \langle x \rangle_\varphi \subseteq I_2$ and $I_1 \cap \langle y \rangle_\varphi \subseteq I_2$. It follows that, $(I_1 \cap \langle x \rangle_\varphi) \vee (I_1 \cap \langle y \rangle_\varphi) \subseteq I_2$. From Theorem 3.2, we deduce that $I_1 \cap (\langle x \rangle_\varphi \vee \langle y \rangle_\varphi) \subseteq I_2$, which implies (by Lemma 2.23 (9)) that, $I_1 \cap \langle x \oplus y \rangle_\varphi \subseteq I_2$. Thus, $x \oplus y \in I_1 \to I_2$.

Suppose that $x \in I_1 \to I_2$ and $y \leq x$. Then, $I_1 \cap \langle x \rangle_\varphi \subseteq I_2$ and $\langle y \rangle_\varphi \overset{Lemma 2.23(9)}{\subseteq} \langle x \rangle_\varphi$. It follows that $I_1 \cap \langle y \rangle_\varphi \subseteq I_1 \cap \langle x \rangle_\varphi \subseteq I_2$. Hence $y \in I_1 \to I_2$. Finally, let $x \in I_1 \to I_2$. Then, $I_1 \cap \langle x \rangle_\varphi \subseteq I_2$. Since, $I_1 \cap \langle \varphi(x) \rangle_\varphi \subseteq I_1 \cap \langle x \rangle_\varphi \subseteq I_2$(due to Lemma 2.23 (6)), we obtain that $\varphi(x) \in I_1 \to I_2$. Therefore, $I_1 \to I_2$ is a state ideal of (L, φ). That is $I_1 \to I_2 \in \mathcal{SI}(L)$.

(2) Now, we prove that $I_1 \cap I \subseteq I_2 \Leftrightarrow I \subseteq I_1 \to I_2$, for any $I, I_1, I_2 \in \mathcal{SI}(L)$. Assume that $I_1 \cap I \subseteq I_2$ and $x \in I$, we have $I_1 \cap \langle x \rangle_\varphi \subseteq I_1 \cap I \subseteq I_2$ (since $\varphi(x) \in I$)). It follows that $x \in I_1 \to I_2$. That is $I \subseteq I_1 \to I_2$. Conversely, let $I \subseteq I_1 \to I_2$ and $x \in I_1 \cap I$. Then we have $x \in I \subseteq I_1 \to I_2$. That is, $I_1 \cap \langle x \rangle_\varphi \subseteq I_2$. Since $x \in I_1 \cap \langle x \rangle_\varphi \subseteq I_2$, we deduce that $x \in I_2$ which implies $I_1 \cap I \subseteq I_2$. Therefore,

$$I_1 \to I_2 = sup\{I \in \mathcal{SI}(L) : I_1 \cap I \subseteq I_2\}$$

and

$$\begin{array}{rcl} I_1 \to_{\_} : \mathcal{SI}(L) & \longrightarrow & \mathcal{SI}(L) \\ I & \longmapsto & (I_1 \to_{\_})(I) = I_1 \to I \end{array}$$

is the right adjoint of $I_1 \cap \_ : \mathcal{SI}(L) \longrightarrow \mathcal{SI}(L)$.

(3) Set $A = \{x \in L : i \wedge n(x \oplus \varphi(x)) \in I_2,\ for\ all\ i \in I_1\ and\ n \in \mathbb{N}^*\}$. First, let $x \in I_1 \to I_2$. Then, $I_1 \cap \langle x \rangle_\varphi \subseteq I_2$. For $n \geq 1$, and $i \in I_1$, we have $n(x \oplus \varphi(x)) \in \langle x \rangle_\varphi$ (since $x, \varphi(x) \in \langle x \rangle_\varphi$)). It follows that, $i \wedge n(x \oplus \varphi(x)) \in I_1 \cap \langle x \rangle_\varphi$, which implies that $i \wedge n(x \oplus \varphi(x)) \in I_2$. Hence $x \in A$.

Conversely, let $x \in A$ and $t \in I_1 \cap \langle x \rangle_\varphi$. Then, there exists $n \in \mathbb{N}^*$ such that $t \leq n(x \oplus \varphi(x))$. Hence, $t = t \wedge n(x \oplus \varphi(x)) \in I_2$. That is, $I_1 \cap \langle x \rangle_\varphi \subseteq I_2$. Hence, $x \in I_1 \to I_2$. Therefore, $I_1 \to I_2 = \{x \in L : i \wedge n(x \oplus \varphi(x)) \in I_2,\ for\ all\ i \in I_1\ and\ n \in \mathbb{N}^*\}$. $\square$

For every $I \in \mathcal{SI}(L)$, we put $I' = I \to \{0\} = \{x \in L : \langle x \rangle_\varphi \cap I = \{0\}\}$. Then from Theorem 3.3 (3), we have the following corollary.

Corollary 3.4. *For every $I \in \mathcal{SI}(L)$, we have $I' = \{x \in L : i \wedge n(x \oplus \varphi(x)) = 0,\ for\ all\ i \in I\ and\ n \in \mathbb{N}^*\}$.*

Theorem 3.5. *Let $I \in \mathcal{SI}(L)$. Then $\mathcal{C}(\mathcal{SI}(L)) = \{\langle x \rangle_\varphi : x \in L\}$.*

Proof. ($\Rightarrow$). Assume that $I \in \mathcal{C}(\mathcal{SI}(L))$. Set $H = \{\langle x \rangle_\varphi : x \in L\}$. Since $I = \bigvee_{x \in I} \langle x \rangle_\varphi$, then there are $\{x_i\}_{1 \leq i \leq n}$ such that $I = \langle x_1 \rangle_\varphi \vee \langle x_2 \rangle_\varphi \vee ... \vee \langle x_n \rangle_\varphi$ (Proposition 2.12). By Lemma 2.23 (9), we have $I = \langle x_1 \oplus x_2 \oplus ... \oplus x_n \rangle_\varphi$. Thus, $I \in H$, that is $\mathcal{C}(\mathcal{SI}(L)) \subseteq H$.
($\Leftarrow$). Let $I \in H$. Then, there exists $x \in L$ such that $I = \langle x \rangle_\varphi$. Assume $\{I_k\}_{k \in K} \subseteq \mathcal{SI}(L)$ and $I = \langle x \rangle_\varphi \subseteq \bigvee_{k \in K} \{I_k\}$. Then, $x \in \bigvee_{k \in K} \{I_k\} = \langle \bigcup_{k \in K} I_k \rangle_\varphi$. It follows that there exist $k_j \in K$, $x_{k_j} \in I_{k_j}$, for all $1 \leq k \leq m$ such that $x \leq x_{k_1} \oplus x_{k_2} \oplus ... \oplus x_{k_m}$. That is, $x \in \langle I_{k_1} \cup I_{k_2} \cup ... \cup I_{k_m} \rangle_\varphi = I_{k_1} \vee I_{k_2} \vee ... \vee I_{k_m}$. $I \in \mathcal{C}(\mathcal{SI}(L))$, that is, $H \subseteq \mathcal{C}(\mathcal{SI}(L))$. Therefore, $\mathcal{C}(\mathcal{SI}(L)) = \{\langle x \rangle_\varphi : x \in L\}$. $\square$

Remark 3.6. Theorem 3.5 means that a state ideal I is a compact element of the frame $\mathcal{SI}(L)$ if and only if it is principal.

Lemma 3.7. *Let L be a residuated lattice and $m, n \in \mathbb{N}^*$, $m, n \geq 2$. For any $x, y \in L$, the following items hold:*

(P16) $x \wedge (ny) \leq n(x'' \wedge y'')$;

(P17) $(nx)'' = nx$;

(P18) $(mx) \wedge (ny) \leq mn(x'' \wedge y'')$.

Proof. (P16). We have

$$x \wedge (ny) = x \wedge \underbrace{(y \oplus ... \oplus y)}_{n\ times} \overset{(P14)}{\leq} \underbrace{(x'' \wedge y'') \oplus ... \oplus (x'' \wedge y'')}_{n\ times} = n(x'' \wedge y'').$$

It follows that $x \wedge (ny) \leq n(x'' \wedge y'')$.

(P17): By induction, we have $2x = x \oplus x = (x' \odot x')'$. It follows that $(2x)'' = (x' \odot x')''' = (x' \odot x')' = 2x$. Let $k \in \mathbb{N}$ such that $k \geq 2$. Assume that $(kx)'' = kx$. Then, we have $(k+1)x = kx \oplus x \overset{hypotesis}{=} (kx)'' \oplus x = ((kx)''' \odot x')' = ((kx)' \odot x')'$. Thus, $((k+1)x)'' = ((kx)' \odot x')''' = ((kx)' \odot x')' = (kx) \oplus x = (k+1)x$. Therefore, $(nx)'' = nx$, for all $n \in \mathbb{N}, n \geq 2$.

(P18). We have $(mx) \wedge (ny) \overset{(P16)}{\leq} n[(mx)'' \wedge y''] \overset{(P17)}{=} n[(mx) \wedge y''] \overset{(P16)}{\leq} n[m(x'' \wedge y'''')] \overset{(RL11)}{=} n[m(x'' \wedge y'')] = mn(x'' \wedge y'')$. Hence, $(mx) \wedge (ny) \leq mn(x'' \wedge y'')$. ☐

Proposition 3.8. *If (L, φ) is a state residuated lattice, then for all $a, b \in L$, we have: $\langle (a \oplus \varphi(a)) \wedge (b \oplus \varphi(b)) \rangle_\varphi = \langle a \rangle_\varphi \cap \langle b \rangle_\varphi$.*

Proof. Let $a, b \in L$. Then from Lemma 2.23 (8), we have $\langle (a \oplus \varphi(a)) \wedge (b \oplus \varphi(b)) \rangle_\varphi \subseteq \langle a \rangle_\varphi \cap \langle b \rangle_\varphi$. Now, let $x \in \langle a \rangle_\varphi \cap \langle b \rangle_\varphi$. Then $x \in \langle a \rangle_\varphi$ and $x \in \langle b \rangle_\varphi$. Hence, from Theorem 2.22 (2), there exist $m, n \geq 1$ such that, $x \leq m(a \oplus \varphi(a))$ and $x \leq n(b \oplus \varphi(b))$. Therefore, $x \leq m(a \oplus \varphi(a)) \wedge n(b \oplus \varphi(b)) \overset{(P18)}{\leq} mn((a \oplus \varphi(a))'' \wedge (b \oplus \varphi(b))'') \overset{(P13)}{=} mn((a \oplus \varphi(a)) \wedge (b \oplus \varphi(b))) \overset{(P3)}{\leq} mn((a \oplus \varphi(a)) \wedge (b \oplus \varphi(b))) \oplus \varphi((a \oplus \varphi(a)) \wedge (b \oplus \varphi(b)))$. That is, $x \in \langle (a \oplus \varphi(a)) \wedge (b \oplus \varphi(b)) \rangle_\varphi$. Hence, $\langle a \rangle_\varphi \cap \langle b \rangle_\varphi \subseteq \langle (a \oplus \varphi(a)) \wedge (b \oplus \varphi(b)) \rangle_\varphi$. Thus, $\langle (a \oplus \varphi(a)) \wedge (b \oplus \varphi(b)) \rangle_\varphi = \langle a \rangle_\varphi \cap \langle b \rangle_\varphi$. ☐

Theorem 3.9. *The lattice $(\mathcal{SI}(L), \subseteq)$ is a coherent frame.*

Proof. (1) From Theorem 3.2, we have that $(\mathcal{SI}(L), \subseteq)$ is a frame.

(2) From Theorem 3.5, we obtain that $\mathcal{C}(\mathcal{SI}(L)) = \{\langle x \rangle_\varphi : x \in L\}$. By Proposition 3.8 we have $\langle x \rangle_\varphi \wedge \langle y \rangle_\varphi = \langle (x \oplus \varphi(x)) \wedge (y \oplus \varphi(y)) \rangle_\varphi$ and by

Lemma 2.23 (9), we have $\langle x\rangle_\varphi \vee \langle y\rangle_\varphi = \langle x\oplus y\rangle_\varphi$ for all $x, y \in L$. Therefore, $(\mathcal{C}(\mathcal{SI}(L)), \subseteq)$ is a sublattice of $(\mathcal{SI}(L), \subseteq)$.

(3) For any $I \in \mathcal{SI}(L)$, we have $I = \bigvee\limits_{x\in I} \langle x\rangle_\varphi$.

(1), (2), (3) combining with Definition 2.13 implies $(\mathcal{SI}(L), \subseteq)$ is a coherent frame. $\square$

The following results are immediate.

Corollary 3.10. *The lattice $(\mathcal{SI}(L), \subseteq)$ is pseudocomplemented. Clearly, for all $I \in \mathcal{SI}(L)$, we have that $I' = I \to \{0\} = \{x \in L : I \cap \langle x\rangle_\varphi = \{0\}\}$ is the pseudocomplement of I*

According to Theorem 2.13 and Corollary 3.10, we have the following Corollary in any residuated lattice L.

Corollary 3.11. 1. *$(\mathcal{I}(L), \subseteq)$ is a coherent frame;*

2. *$(\mathcal{I}(L), \subseteq)$ is a pseudocomplemented lattice.*

We recall that a Heyting algebra [1] is a lattice $(L, \vee, \wedge)$ with 0 such that for every $x, y \in L$, there exists an element $x \to y \in L$ (call the pseudocomplement of x with respect to y) such that for every $z \in L, x \wedge z \leq y \Leftrightarrow z \leq x \to y$ (that is, $x \to y = sup\{z \in L : x \wedge z \leq y\}$).

Remark 3.12. From Theorem 3.3, $(\mathcal{SI}(L), \vee, \wedge, ', \{0\})$ is a Heyting algebra, where for $I \in \mathcal{SI}(L)$, $I' = I \to \{0\} = \{x \in L : \langle x\rangle_\varphi \cap I = \{0\}\}$.

Lemma 3.13. *Let L be a residuated lattice. Then, for any $x, y, z \in L$ the following hold:*
(P19) $x \wedge (y \oplus z) \leq x'' \wedge (y \oplus z)'' \leq (x'' \wedge y'') \oplus (x'' \wedge z'')$.

Proof. Let $x, y, z \in L$.

From (RL9), we have $x \leq x''$ and $y\oplus z \leq (y\oplus z)''$. Hence, $x\wedge(y\oplus z) \leq x''\wedge (y\oplus z)''$. In addition, $x''\wedge(y\oplus z)'' = x''\wedge(y'\odot z')''' \overset{(RL11)}{=} x''\wedge(y'\odot z')' \overset{(RL7)}{=} [x'\vee(y'\odot z')]' \overset{(RL4),(RL16)}{\leq} [(x'\vee y')\odot(x'\vee z')]' \overset{(RL11)}{=} [(x'\vee y')\odot(x'\vee z')]''' \overset{(RL7)}{=} [(x'\vee y')''\odot(x'\vee z')'']' \overset{(RL7)}{=} (x''\wedge y'')'\odot(x''\wedge z'')']' = (x''\wedge y'')\oplus(x''\wedge z'')$.

Thus, $x''\wedge(y\oplus z)'' \leq (x''\wedge y'')\oplus(x''\wedge z'')$. Therefore, $x\wedge(y\oplus z) \leq x''\wedge(y\oplus z)'' \leq (x''\wedge y'')\oplus(x''\wedge z'')$.

$\square$

Proposition 3.14. *For any $a, b \in L$, we have:*

1. $(\langle a\rangle_\varphi)' = \{x \in L : (a \oplus \varphi(a)) \wedge (x \oplus \varphi(x)) = 0\}$;
2. $(\langle a\rangle_\varphi)' \cap (\langle b\rangle_\varphi)' = (\langle a \oplus b\rangle_\varphi)'$.

Proof. (1) We have, $(\langle a\rangle_\varphi)' = \langle a\rangle_\varphi \to \{0\} = \{x \in L : \langle a\rangle_\varphi \cap \langle x\rangle_\varphi = \{0\}\} \overset{Proposition\ 3.8}{=} \{x \in L : \langle (a \oplus \varphi(a)) \wedge (x \oplus \varphi(x))\rangle_\varphi = \{0\}\} = \{x \in L : (a \oplus \varphi(a)) \wedge (x \oplus \varphi(x)) = 0\}$.

(2) Let $x \in (\langle a\rangle_\varphi)' \cap (\langle b\rangle_\varphi)'$. Then by (1), we have $(a \oplus \varphi(a)) \wedge (x \oplus \varphi(x)) = 0$ and $(b \oplus \varphi(b)) \wedge (x \oplus \varphi(x)) = 0$. Furthermore by (P19), we have $((a \oplus \varphi(a)) \oplus (b \oplus \varphi(b))) \wedge (x \oplus \varphi(x)) \leq ((a \oplus \varphi(a))'' \wedge (x \oplus \varphi(x))'') \oplus ((b \oplus \varphi(b))'' \wedge (x \oplus \varphi(x))'') \overset{(P13)}{=} ((a \oplus \varphi(a)) \wedge (x \oplus \varphi(x))) \oplus ((b \oplus \varphi(b)) \wedge (x \oplus \varphi(x))) = 0$, hence $((a \oplus \varphi(a)) \oplus (b \oplus \varphi(b))) \wedge (x \oplus \varphi(x)) = 0$. Then combining (P19) and (P13), we have $(2((a \oplus \varphi(a)) \oplus (b \oplus \varphi(b)))) \wedge (x \oplus \varphi(x)) \leq 2(((a \oplus \varphi(a)) \oplus (b \oplus \varphi(b))) \wedge (x \oplus \varphi(x))) = 0$. Moreover, by (RL6) and (SO22), $2(((a \oplus \varphi(a)) \oplus (b \oplus \varphi(b))) = ((a \oplus \varphi(a)) \oplus (b \oplus \varphi(b))) \oplus ((a \oplus \varphi(a)) \oplus (b \oplus \varphi(b))) \geq a \oplus b \oplus \varphi(a) \oplus \varphi(b) \geq a \oplus b \oplus \varphi(a \oplus b)$. Hence, $(a \oplus b \oplus \varphi(a \oplus b)) \wedge (x \oplus \varphi(x)) = 0$. Then by (1), we have $x \in (\langle a \oplus b\rangle_\varphi)'$. Therefore $(\langle a\rangle_\varphi)' \cap (\langle b\rangle_\varphi)' \subseteq (\langle a \oplus b\rangle_\varphi)'$.

Conversely, let $x \in (\langle a \oplus b\rangle_\varphi)'$. Then by (1), $((a \oplus b) \oplus \varphi(a \oplus b))) \wedge (x \oplus \varphi(x)) = 0$, we have $a \leq a \oplus b$, so $\varphi(a) \leq \varphi(a \oplus b)$. It follows that $a \oplus \varphi(a) \leq (a \oplus b) \oplus \varphi(a \oplus b)$. We obtain $(a \oplus \varphi(a)) \wedge (x \oplus \varphi(x)) \leq ((a \oplus b) \oplus \varphi(a \oplus b))) \wedge (x \oplus \varphi(x)) = 0$ and thus, $(a \oplus \varphi(a)) \wedge (x \oplus \varphi(x)) = 0$. Analogously, $(b \oplus \varphi(b)) \wedge (x \oplus \varphi(x)) = 0$. Therefore, $x \in (\langle a\rangle_\varphi)' \cap (\langle b\rangle_\varphi)'$. ☐

Lemma 3.15. *Let $x \in L$. Then, the following holds:*

(P20) $x'' \oplus \varphi(x'') = x \oplus \varphi(x)$.

Proof. We have $x'' \oplus \varphi(x'') \overset{(SO21)}{=} x'' \oplus (\varphi(x))'' = (x''' \odot (\varphi(x))''')' \overset{(RL11)}{=} (x' \odot (\varphi(x))')' = x \oplus \varphi(x)$. ☐

Theorem 3.16. *In a state residuated lattice (L, φ), the following are equivalent:*

(i) $(\mathcal{SI}(L), \vee, \wedge, ', \{0\}, L)$ *is a Boolean algebra;*

(ii) *Every state ideal of* (L,φ) *is principal and for every* $x \in L$*, there is* $n \in \mathbb{N}^*$*, such that* $(x\oplus\varphi(x))\wedge((n(x\oplus\varphi(x)))'\oplus\varphi((n(x\oplus\varphi(x)))'))=0$.

Proof. (i)$\Rightarrow$(ii). Assume that $(\mathcal{SI}(L),\vee,\wedge,',\{0\},L)$ is a Boolean algebra. Then for every $I\in\mathcal{SI}(L)$, $I\vee I'=L$, thus, $1\in I\vee I'$. But according to Theorem 2.22 (4), $I\vee I':=\langle I\cup I'\rangle_\varphi=\{x\in L: x\leq y\oplus z, \textit{with } y\in I \textit{ and } z\in I'\}$. Hence, there are $y\in I$ and $z\in I'$ such that $y\oplus z=1$. We will prove that $I=\langle y\rangle_\varphi$. Since $y\in I$, it follows that $\langle y\rangle_\varphi\subseteq I$. According to Corollary 3.4, $I'=\{a\in L: x\wedge n(a\oplus\varphi(a))=0, \textit{ for all } x\in I \textit{ and } n\in\mathbb{N}^*\}$. Thus, $x\wedge n(z\oplus\varphi(z))=0$, for all $x\in I$ and $n\in\mathbb{N}^*$. Then $x\odot z\leq x\wedge z=0$ for every $x\in I$, that is $x\odot z=0$, for every $x\in I$. Hence, $x''\to z'\overset{(RL11)}{=}(x\odot z)'=1$, that is $x''\leq z'$. Since $y\oplus z=1$, we obtain that $n(y\oplus\varphi(y))\oplus z=1$, for every $n\in\mathbb{N}^*$ (because $y\oplus z\leq n(y\oplus\varphi(y))\oplus z$). Hence by (P1), $(n(y\oplus\varphi(y)))'\to z''=1$, that is $(n(y\oplus\varphi(y)))'\leq z''$. It follows that $z'\overset{(RL11)}{=}z'''\overset{(RL4)}{\leq}(n(y\oplus\varphi(y)))''\overset{(P17)}{=}n(y\oplus\varphi(y))$. Hence $z'\leq n(y\oplus\varphi(y))$. Thus, we obtain that $x\overset{(RL9)}{\leq}x''\leq z'\leq n(y\oplus\varphi(y))$, i.e., $x\leq n(y\oplus\varphi(y))$, for every $x\in I$, that is, $I\subseteq\langle y\rangle_\varphi$. Therefore, $I=\langle y\rangle_\varphi$.

Now, let $x\in L$. Since $(\mathcal{SI}(L),\vee,\wedge,',\{0\},L)$ is a Boolean algebra, we have $L=\langle x\rangle_\varphi\vee(\langle x\rangle_\varphi)'=\{t\in L: t\leq y\oplus n(x\oplus\varphi(x)), \textit{ for some } n\in\mathbb{N}^*,\ y\in(\langle x\rangle_\varphi)'\}$. Hence, there exist $y\in(\langle x\rangle_\varphi)'$ and $n\in\mathbb{N}^*$, such that $y\oplus n(x\oplus\varphi(x))=1$. Since $y\in(\langle x\rangle_\varphi)'$, then by Proposition 3.14(1), $(x\oplus\varphi(x))\wedge(y\oplus\varphi(y))=0$. From $y\oplus n(x\oplus\varphi(x))=1$, we deduce that $y'\to(n(x\oplus\varphi(x)))''=1$ which implies $y'\leq(n(x\oplus\varphi(x)))''\overset{(P17)}{=}n(x\oplus\varphi(x))$. Thus by (RL4), $(n(x\oplus\varphi(x)))'\leq y''$, which implies $\varphi((n(x\oplus\varphi(x)))')\overset{(SO10)}{\leq}\varphi(y'')$. Hence $(n(x\oplus\varphi(x)))'\oplus\varphi((n(x\oplus\varphi(x)))')\overset{(P6)}{\leq}y''\oplus\varphi(y'')\overset{(P20)}{=}y\oplus\varphi(y)$. It follows that, $(x\oplus\varphi(x))\wedge((n(x\oplus\varphi(x)))'\oplus\varphi((n(x\oplus\varphi(x)))'))\leq(x\oplus\varphi(x))\wedge(y\oplus\varphi(y))=0$. Therefore, $(x\oplus\varphi(x))\wedge((n(x\oplus\varphi(x)))'\oplus\varphi((n(x\oplus\varphi(x)))'))=0$.

(ii)$\Rightarrow$(i). By Remark 3.12, $(\mathcal{SI}(L),\vee,\wedge,',\{0\})$ is a Heyting algebra. In order to prove that $(\mathcal{SI}(L),\vee,\wedge,',\{0\},L)$ is a Boolean algebra, it is enough to prove that for every $I\in\mathcal{SI}(L)$, we have $I'=\{0\}\Leftrightarrow I=L$ (according to [1]). Let $I\in\mathcal{SI}(L)$ with $I'=\{0\}$. By the hypothesis, every state ideal is principal. Hence, there is $x\in L$ such that $I=\langle x\rangle_\varphi$. Thus, $(\langle x\rangle_\varphi)'=\{0\}$. Moreover, there is $n\in\mathbb{N}^*$ such that $(x\oplus\varphi(x))\wedge$

$((n(x \oplus \varphi(x)))' \oplus \varphi((n(x \oplus \varphi(x)))')) = 0$. By proposition 3.14 (1), it follows that $(n(x \oplus \varphi(x)))' \in (\langle x \rangle_\varphi)' = \{0\}$. Thus, $(n(x \oplus \varphi(x)))' = 0$, that is $n(x \oplus \varphi(x)) \overset{(P17)}{=} (n(x \oplus \varphi(x)))'' = 1$. Since $n(x \oplus \varphi(x)) \in \langle x \rangle_\varphi$, we deduce that $1 \in \langle x \rangle_\varphi = I$. Therefore, $I = \langle x \rangle_\varphi = L$. □

4 Prime state ideals in state residuated lattices

In this section, we introduce the notion of prime state ideals in a state residuated lattice, illustrate with some examples and lay out some characterizations of prime state ideals. The prime state ideal theorem is set forth.

Definition 4.1. Let P be a proper state ideal of (L, φ). P is said to be *prime* if for all $P_1, P_2 \in \mathcal{SI}(L)$, $P = P_1 \cap P_2$ implies $P = P_1$ or $P = P_2$.

The following result is a characterization of a prime state ideal.

Proposition 4.2. *Let P be a proper state ideal of (L, φ). Then the following are equivalent:*

1. *P is prime*
2. *For all $P_1, P_2 \in \mathcal{SI}(L)$, if $P_1 \cap P_2 \subseteq P$, then $P_1 \subseteq P$ or $P_2 = P$;*
3. *For all $a, b \in L$, if $(a \oplus \varphi(a)) \wedge (b \oplus \varphi(b)) \in P$, then $a \in P$ or $b \in P$.*

Proof. (1) $\Rightarrow$ (2) Let $P_1, P_2 \in \mathcal{SI}(L)$ such that $P_1 \cap P_2 \subseteq P$. Then, $(P_1 \cap P_2) \vee P = P$. From Proposition 3.1, the lattice $(\mathcal{SI}(L), \subseteq)$ is Brouwerian, so it is distributive. It follows that $(P_1 \vee P) \cap (P_2 \vee P) = P$. Now by (1), P is prime so, $P_1 \vee P = P$ or $P_2 \vee P = P$. Thus, $P_1 \subseteq P$ or $P_2 \subseteq P$.

(2) $\Rightarrow$ (1) Let $P_1, P_2 \in \mathcal{SI}(L)$ such that $P = P_1 \cap P_2$. Then, $P_1 \cap P_2 \subseteq P$. By the hypothesis, $P_1 \subseteq P$ or $P_2 = P$. If $P_1 \subseteq P$, then, $P = P_1 \cap P_2 \subseteq P_1 \subseteq P$. Hence, $P_1 = P$. By similar way, if $P_2 \subseteq P$, we get $P_2 = P$.

(1) $\Rightarrow$ (3). Let $a, b \in L$. Suppose that $(a \oplus \varphi(a)) \wedge (b \oplus \varphi(b)) \in P$. Set $P_1 = \langle P, a \rangle_\varphi$ and $P_2 = \langle P, b \rangle_\varphi$. Obviously, $P \subseteq P_1 \cap P_2$. Let $x \in P_1 \cap P_2$, then by Theorem 2.22 (3), there are $l, k \in P$ and $m, n \geq 1$ such that $x \leq k \oplus m(a \oplus \varphi(a))$ and $x \leq l \oplus n(b \oplus \varphi(b))$. Then by the property of joins, we have

$x \leq (k \oplus m(a \oplus \varphi(a))) \wedge (l \oplus n(b \oplus \varphi(b))) \overset{(P19)}{\leq} ((k \oplus m(a \oplus \varphi(a)))'' \wedge l'') \oplus ((k \oplus m(a \oplus \varphi(a)))'' \wedge (n(b \oplus \varphi(b)))'') \overset{(P13),(P17)}{\leq} ((k \oplus m(a \oplus \varphi(a))) \wedge l'') \oplus ((k \oplus m(a \oplus \varphi(a))) \wedge n(b \oplus \varphi(b))) \overset{(P19)}{\leq} (k'' \wedge l'''') \oplus (m(a \oplus \varphi(a))'' \wedge l'''') \oplus (k'' \wedge (n(b \oplus \varphi(b)))'' \oplus (m(a \oplus \varphi(a))'' \wedge (n(b \oplus \varphi(b)))'' \overset{(P13),(RL11)}{\leq} (k'' \wedge l'') \oplus (m(a \oplus \varphi(a)) \wedge l'') \oplus (k'' \wedge (n(b \oplus \varphi(b))) \oplus (m(a \oplus \varphi(a)) \wedge (n(b \oplus \varphi(b))) \overset{(P18)}{\leq} (k'' \wedge l'') \oplus (m(a \oplus \varphi(a)) \wedge l'') \oplus (k'' \wedge (n(b \oplus \varphi(b))) \oplus mn((a \oplus \varphi(a))'' \wedge (b \oplus \varphi(b))'') \overset{(P19)}{\leq} (k'' \wedge l'') \oplus (m(a \oplus \varphi(a)) \wedge l'') \oplus (k'' \wedge (n(b \oplus \varphi(b))) \oplus mn((a \oplus \varphi(a)) \wedge (b \oplus \varphi(b)))$.

But $(k'' \wedge l''), (l'' \wedge m(a \oplus \varphi(a))), (k'' \wedge n(b \oplus \varphi(b))), mn((a \oplus \varphi(a)) \wedge (b \oplus \varphi(b))) \in P$. Thus $x \in P$. Hence $P = P_1 \cap P_2$. Therefore by (1), $P = P_1$ or $P = P_2$, that is, $a \in P$ or $b \in P$.

(3) $\Rightarrow$ (1) Let $P_1, P_2 \in \mathcal{SI}(L)$ such that $P = P_1 \cap P_2$. Suppose that $P \neq P_1$ and $P \neq P_2$ and let $a \in P_1 \backslash P$ and $b \in P_2 \backslash P$. Then, $(a \oplus \varphi(a)) \wedge (b \oplus \varphi(b)) \in P_1 \cap P_2 = P$, that is a contradiction. Thus $P = P_1$ or $P = P_2$. Therefore, P is a prime state ideal of (L, φ). $\square$

We denote the set of all prime state ideals of (L, φ) by $Spect_{\varphi}(L)$.

Example 4.3. Let $L = [0, 1]$ be the unit interval. Define the algebraic structure $L = (L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ such that for all $x, y \in I$,
$x \wedge y = min\{x, y\}$, $x \vee y = max\{x, y\}$, $x \odot y = \begin{cases} 0, & \text{if x + y} \leq \frac{1}{2} \\ min\{x, y\}, & \text{othewise.} \end{cases}$

and $x \rightarrow y = \begin{cases} 1, & \text{if x } \leq \text{y} \\ max\{\frac{1}{2} - x, y\}, & \text{othewise.} \end{cases}$

Then, L is a residuated lattice [28]. From Remark 2.19, (L, id_L) is a state residuated lattice.

One can check that the subsets $L, I_a = [0, a], J_a = [0, a), J = [0, \frac{1}{4})$ and $I = \{0\}$ (with $a \in (0, \frac{1}{4})$) are the only ideals of L. Therefore, they are state ideals of (L, id_L).
$J = [0, \frac{1}{4})$ is a prime state ideal of (L, id_L).

The following example gives a prime state ideal of a state residuated lattice which is not a prime ideal.

Example 4.4. Let $L = \{0, a, b, c, d, 1\}$ be a poset with Hasse digram and Cayley tables as follows:

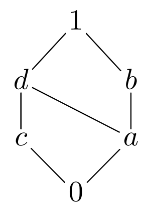

Figure 1: Hasse diagram of a SRL that admits a prime state ideal which is not a prime ideal.

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	d	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	a	b	b	d	1	1
1	0	a	b	c	d	1

Table 1: Cayley tables of the operations $\odot$ and $\to$ of a SRL that admits a prime state ideal which is not a prime ideal.

Then $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a residuated lattice. The ideals of L are $\{0\}$,$\{0, c\}$, $\{0, a, b\}$ and L. Now we define a map τ on L as follows:

$$\varphi(x) = \begin{cases} 0, & \text{if x} \in \{0, a, b\}; \\ 1, & \text{if x} \in \{c, d, 1\}. \end{cases}$$

One can easily check that (L, φ) is a state residuated lattice. The ideals of L are $\{0\}$,$\{0, c\}$, $\{0, a, b\}$ and L. In addition, the state ideals of (L, φ) are $\{0\}$, $\{0, a, b\}$ and L. We have $\{0\} = \{0, c\} \cap \{0, a, b\}$ but $\{0\} \neq \{0, c\}$ and $\{0\} \neq \{0, a, b\}$, so the ideal $\{0\}$ is not a prime ideal of L. However, still as a state ideal (L, φ) and according to Proposition 4.2 (1), $\{0\}$ is a prime state ideal of (L, φ).

Now, we establish our main theorem.

Theorem 4.5. *(prime state ideal theorem)*

Let F be a filter in the lattice $(L, \subseteq)$, and I be a state ideal of (L, φ) such that $I \cap F = \emptyset$. Then, there is a prime state ideal P of (L, φ) such that $I \subseteq P$ and $P \cap F = \emptyset$.

Proof. Let us consider the set $k(I) = \{J : J \in \mathcal{SI}(L), I \subseteq J \text{ and } J \cap F = \emptyset\}$. We have $I \in \mathcal{SI}(L), I \subseteq I$ and $I \cap F = \emptyset$. Then $I \in k(I)$, that is $k(I) \neq \emptyset$. On can easily prove that the set $k(I)$ is inductively ordered by inclusion and, by Zorn's lemma, it has a maximal element P. We will prove that $P \in Spect_{\varphi}(L)$. Since $P \in k(I)$, we deduce that P is a proper state ideal and $P \cap F = \emptyset$. Let $a, b \in L$ such that $(a \oplus \varphi(a)) \wedge (b \oplus \varphi(b)) \in P$. Suppose that $a \notin P$ and $b \notin P$ and consider the sets $\langle P, a\rangle_{\varphi}$ and $\langle P, b\rangle_{\varphi}$. Then, P is strictly contained in $\langle P, a\rangle_{\varphi}$ and $\langle P, b\rangle_{\varphi}$ and the maximality of P implies that $\langle P, a\rangle_{\varphi} \notin k(I)$ and $\langle P, b\rangle_{\varphi} \notin k(I)$. Thus, $\langle P, a\rangle_{\varphi} \cap F \neq \emptyset$ and $\langle P, b\rangle_{\varphi} \cap F \neq \emptyset$. Let $x \in \langle P, a\rangle_{\varphi} \cap F$ and $y \in \langle P, b\rangle_{\varphi} \cap F \neq \emptyset$. According to Theorem 2.22 (3), there are $k, l \in P$ and $m, n \geq 1$ such that $x \leq k \oplus m(a \oplus \varphi(a))$ and $y \leq l \oplus n(b \oplus \varphi(b))$. Then,

$$x \wedge y \leq (k \oplus m(a \oplus \varphi(a))) \wedge (l \oplus n(b \oplus \varphi(b))) \overset{(P19)}{\leq} ((k \oplus m(a \oplus \varphi(a)))'' \wedge l'') \oplus ((k \oplus m(a \oplus \varphi(a)))'' \wedge (n(b \oplus \varphi(b)))'') \overset{(P13),(P17)}{\leq} ((k \oplus m(a \oplus \varphi(a))) \wedge l'') \oplus ((k \oplus m(a \oplus \varphi(a))) \wedge n(b \oplus \varphi(b))) \overset{(P19)}{\leq} (k'' \wedge l'''') \oplus (m(a \oplus \varphi(a))'' \wedge l'''') \oplus (k'' \wedge (n(b \oplus \varphi(b)))'' \oplus (m(a \oplus \varphi(a))'' \wedge (n(b \oplus \varphi(b)))'' \overset{(P13),(RL11)}{\leq} (k'' \wedge l'') \oplus (m(a \oplus \varphi(a)) \wedge l'') \oplus (k'' \wedge (n(b \oplus \varphi(b))) \oplus (m(a \oplus \varphi(a)) \wedge (n(b \oplus \varphi(b))) \overset{(P18)}{\leq} (k'' \wedge l'') \oplus (m(a \oplus \varphi(a)) \wedge l'') \oplus (k'' \wedge (n(b \oplus \varphi(b))) \oplus mn((a \oplus \varphi(a))'' \wedge (b \oplus \varphi(b))'') \overset{(P19)}{\leq} (k'' \wedge l'') \oplus (m(a \oplus \varphi(a)) \wedge l'') \oplus (k'' \wedge (n(b \oplus \varphi(b))) \oplus mn((a \oplus \varphi(a)) \wedge (b \oplus \varphi(b))).$$

But $(k'' \wedge l''), (l'' \wedge m(a \oplus \varphi(a))), (k'' \wedge n(b \oplus \varphi(b))), mn((a \oplus \varphi(a)) \wedge (b \oplus \varphi(b))) \in P$. Hence $x \wedge y \in P$. On the other hand, since F is a filter of the lattice $(L, \subseteq)$, we deduce that $x \wedge y \in F$, and therefore $P \cap F \neq \emptyset$, which is a contradiction. Thus $P \in Spect_{\varphi}(L)$. $\square$

Proposition 4.6. *Let I be a proper state ideal of (L, φ). Then, there is a maximal state ideal I_0 of (L, φ) such that $I \subseteq I_0$.*

Proof. Let us consider the set

$$L_I = \{J : J \textit{ is a proper state ideal containing } I\}.$$

We have I is a proper state ideal of (L, φ) and $I \subseteq I$. So $I \in L_I$, that is $L_I \neq \emptyset$. On can prove that L_I is inductively ordered by inclusion. So, by Zorn's lemma, L_I has a maximal element I_0. We are going to prove that I_0 is a maximal state ideal of (L, φ). Indeed, if I_1 is a proper state ideal of (L, φ) such that $I_0 \subseteq I_1$, then $I_1 \in L_I$ and the maximality of I_0 implies that $I_1 = I_0$ □

Proposition 4.7. *Let $a \in L$, $a > 0$. Then, there is a prime state ideal P of (L, φ) so that $a \notin P$.*

Proof. Since $\{0\}$ is a state ideal and $\{0\} \cap [a) = \emptyset$ (where $[a)$ is the filter generated by $\{a\}$ in the lattice $(L, \subseteq)$). Hence by Theorem 4.5, there exists a prime state ideal P such that $P \cap \langle a \rangle = \emptyset$. Thus $a \notin P$. □

Conclusion

This work was devoted to the prime state ideals theorem in the framework of state residuated lattices. We have investigated the lattice of state ideals $\mathcal{SI}(L)$ of a SRL and obtained that it is a coherent frame. Moreover, we have described the set of compact elements of the sublattice $\mathcal{SI}(L)$. Furthermore, We have characterized the SRL for which the lattice $\mathcal{SI}(L)$ is a Boolean algebra. In addition, we have proved the prime state ideal theorem and gave some related properties. In the same view as the work in [16], we will study the lattice of L-fuzzy state ideals in state residuated lattices. Analogously to the papers [6, 19, 27] , another direction will consist of investigating other types of state ideals namely obstinate, maximal, Boolean, primary, implicative and integral state ideals in state residuated lattices.

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Francis Woumfo *Department of Mathematics and Computer Science, Faculty of Science, University of Dschang, P.O. Box 67, Dschang, Cameroon.*

Email: fwoumfo@yahoo.fr

Etienne Romuald Temgoua Alomo *Department of Mathematics, École Normale Supérieure*

de Yaoundé, University of Yaoundé 1, P.O. Box 47, Yaoundé, Cameroon.
Email: retemgoua@yahoo.fr

Celestin Lele *Department of Mathematics and Computer Science, Faculty of Science, University of Dschang, P.O. Box 67, Dschang, Cameroon.*
Email: celestinlele@yahoo.com